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# AN AXIOMATIZATION OF THE LEONTIEF PREFERENCES

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*An axiomatic characterization of the well known Leontief preferences is given. The key axiom is upper consistency, which states that for any two bundles, another bundle is weakly preferred to at least one of them if and only if it is weakly preferred to the bundle that contains the least amount of each commodity in them. (JEL: D01, D11)*

## 1. Introduction

In economic literature, it is often assumed that a decision maker, either as an individual or as a group, has the following preferences: alternative $x \in \mathbf{R}_+^n$ is weakly preferred to alternative $y \in \mathbf{R}_+^n$ if and only if $\min_i \{a_i x_i\} \geq \min_i \{a_i y_i\}$, where $x, y \in \mathbf{R}_+^n$ can be either consumption bundles, as in the consumer choice literature, or utility profiles as in the social choice literature. Such preferences are known as the Leontief preferences.

The Leontief preferences are representable by the Leontief utility function, which is one of the standard functional forms used in economics. Moreover, in consumer choice theory, it is the most useful tool to demonstrate the idea of complementarity of economic goods, often attributed to the case of right and left shoes. We do not have any clear reference on its entrance to this field. However, its analogy in production theory, the Leontief production technology used in input-output analysis was developed as early as 1933 by Wassily Leontief (see Dorfman (2008)). The standard reference in this respect is Leontief (1951) (see for instance, Chap. 9 in Dorfman, Samuelson and Solow (1958)).

This paper contributes to the economic literature on choice theory by axiomatizing these prototypical preferences. In particular, we give three basic axioms that fully characterize weak

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preferences on $\mathbf{R}^n_+$ as the Leontief preferences (Theorem 1 in Section 3). Some of the earlier works related to ours are as follows. Segal and Sobel (2002) provides a joint axiomatization of *min*, *max* and *sum* utility functions, defined respectively as, for all $x \in \mathbf{R}^n$ with $n \geq 3$, $u(x) = \min_i \{x_i\}$, $u(x) = \max_i \{x_i\}$ and $u(x) = \sum x_i$, with five axioms: continuity, monotonicity, symmetry, linearity and partial separability (see Theorem 2 in Segal and Sobel (2002)). However, since they do not directly characterize the Leontief utility function, which corresponds to $u(x) = \min_i \{x_i\}$, one needs additional axiom(s) in order to obtain such characterization from their result. Moreover, they only consider the unweighted *min*, *max* and *sum* utility functions whereas the standard form of the Leontief utility function involves positive weights.

In the social choice literature, the *maximin* social welfare ordering defined on the utility profiles of the society members has the same form as the Leontief preferences. It can be defined for societies with finite or countably infinite members: see for instance Bosmans and Ooghe (2006) and Miyagishima (2010) for the former, and Lauwers (1997) and Chambers (2009) for the latter. The former is more relevant to us since our setting is restricted to a finite dimensional Euclidean space. In that case, Bosmans and Ooghe (2006) characterizes the *maximin* social welfare ordering with four axioms: anonymity, continuity, weak Pareto and Hammond equity, and Miyagishima (2010) shows that one can drop anonymity and modify Hammond equity into a weighted Hammond equity to characterize the *weighted maximin* social welfare ordering. In contrast to these characterizations, we do not use any of the continuity, weak Pareto and Hammond equity, but only use a counterpart of anonymity (see **A.2** in Section 2) in our characterization. The next section introduces the main definitions, Section 3 gives the main result, the characterization theorem, and the last section concludes.

## 2. The preliminaries

Define preferences on a set $X$ in terms of a binary relation $\succsim$ ("weakly preferred to") which is:

**complete:** for all $x, y \in X$, $x \succsim y$, $y \succsim x$ or both;
**transitive:** for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

We call $\succsim$ as a weak order on $X$. As usual, $x \succ y$ means $x \succsim y$, but not $y \succsim x$, whereas $x \sim y$ means both $x \succsim y$ and $y \succsim x$. For any $x \in X$, let $\mathbf{U}(x) = \{y \in X : y \succsim x\} \subseteq X$ be the set of alternatives that are at least as good as $x \in X$, and $\mathbf{I}(x) = \{y \in X : y \sim x\} \subseteq X$ be the set of alternatives that are indifferent to $x \in X$, according to $\succsim$. From now on we take $X$ as $\mathbf{R}^n_+$. A weak order $\succsim$ on $\mathbf{R}^n_+$ is the Leontief preferences if

$$\forall x, y \in \mathbf{R}^n_+,\ x \succsim y$$
$$\Leftrightarrow \min\{x_1, \ldots, x_n\} \geq \min\{y_1, \ldots, y_n\},$$

and it is an $\mathbf{a} = (a_1, \ldots, a_n)$-weighted Leontief preferences with $a_i > 0$ for $i = 1, \ldots, n$, if

$$\forall x, y \in \mathbf{R}^n_+,\ x \succsim y$$
$$\Leftrightarrow \min\{a_1 x_1, \ldots, a_n x_n\} \geq \min\{a_1 y_1, \ldots, a_n y_n\}.$$

Note that the former is a special case of the latter with $a_i = 1$, $i = 1, \ldots, n$. For any $x, y \in \mathbf{R}^n_+$, let $\min\{x, y\} = (\min\{x_1, y_1\}, \ldots, \min\{x_n, y_n\}) \in \mathbf{R}^n_+$. For any $\varepsilon > 0$, an $\varepsilon$-ball around $x \in \mathbf{R}^n_+$ is $B_\varepsilon(x) = \{y \in \mathbf{R}^n_+ : \sqrt{\sum_{i=1}^n (x_i - y_i)^2} < \varepsilon\}$. For any positive numbers $a_1, \ldots, a_n$, let $l(a_1, \ldots, a_n) \subset \mathbf{R}^n_+$ be a line with

$$l(a_1, \ldots, a_n) = \{x \in \mathbf{R}^n_+ : a_1 x_1 = \ldots = a_n x_n\}$$

and when $a_i = 1$ for $i = 1, \ldots, n$, we write $l$ instead of $l(1, \ldots, 1)$. Finally, for any $x \in \mathbf{R}^n_+$ and for $i, j = 1, \ldots, n$, let $\pi_{i,j}(x) \in \mathbf{R}^n_+$ be a vector obtained from $x \in \mathbf{R}^n_+$ by interchanging its *i'th* and *j'th* components, and for any $x, y \in \mathbf{R}^n_+$ let $x * y \in \mathbf{R}^n_+$ be defined as $x * y = (x_1 y_1, \ldots, x_n y_n)$.

We say that $\succsim$ on $\mathbf{R}^n_+$ is

**A.1:** *Upper consistent* if $\forall x, y \in \mathbf{R}^n_+, \mathbf{U}(\min\{x, y\}) = \mathbf{U}(x) \cup \mathbf{U}(y)$.

**A.2:** *Symmetric* (or *Neutral*) with respect to $l$ if whenever $x \sim y$, we have $\pi_{i,j}(x) \sim \pi_{i,j}(y)$, for all $i, j = 1, \ldots, n$.

**A.2':** *Symmetric* (or *Neutral*) with respect to $l(a_1, \ldots, a_n)$ if whenever $x \sim y$, we have

$$(\frac{1}{a_1},\ldots,\frac{1}{a_n})*\pi_{i,j}(a_1x_1,\ldots,a_nx_n) \sim (\frac{1}{a_1},\ldots,\frac{1}{a_n})*\pi_{i,j}(a_1y_1,\ldots,a_ny_n)$$

for all $i, j = 1,\ldots,n.$

**A.3:** *Locally non-satiable* if $\forall\varepsilon > 0$ and $\forall x\in\mathbf{R}^n_+, \exists y\in B_\varepsilon(x)$ with $y \succ x$.

**A.1** says that for any two bundles, another bundle is weakly preferred to at least one of them if and only if it is weakly preferred to the bundle that contains the least amount of each commodity in them. **A.2** is a way of saying that goods are of equal importance, i.e. it does not matter if one exchanges the roles of right and left shoes. More precisely, the indifference relation induced by $\succsim$ is unaffected by renaming of the commodities. **A.2'** is a variant of **A.2** after rescaling of coordinates with $\mathbf{a}\in\mathbf{R}^n_{++}$. **A.3** is a standard axiom in microeconomics and it rules out thick indifference curves.

## *3. The characterization theorem*

We now state and prove a characterization theorem for $\mathbf{R}^2_+$ since the main idea is best illustrated in that case. However, we remark here that the result holds in the general domain of $\mathbf{R}^n_+$ (see Theorem 3 in Appendix A).

**Theorem 1.** *Let $\succsim$ be a weak order on $\mathbf{R}^2_+$. Then,*

**(a)** $\succsim$ *satisfies* **A.1**, **A.2** *and* **A.3** *if and only if it is the Leontief preferences, and*

**(b)** $\succsim$ *satisfies* **A.1**, **A.2'** *and* **A.3** *if and only if it is an* $(a_1, a_2)$*-weighted Leontief preferences.*

**Proof.** Since IF parts are easy to check, we prove ONLY IF parts.

**(a)** Suppose $\succsim$ satisfies **A.1–A.3**. Consider $x^*\in l$, i.e. $x^*_1 = x^*_2$. Let

$$L(x^*) = \{x\in\mathbf{R}^2_+ : x_i \geq x^*_i, i = 1,2\}.$$

We proceed in 3 steps.

**Step 1:** We claim that $\mathbf{U}(x^*) = L(x^*)$. First, note that for any $x\in L(x^*)$, $\min\{x^*, x\} = x^*$ and then by **A.1**, $\mathbf{U}(x)\subseteq\mathbf{U}(x^*)$. In particular, $x\in\mathbf{U}(x^*)$. Hence, $L(x^*)\subseteq\mathbf{U}(x^*)$. For the other inclusion, suppose $\exists y\in\mathbf{U}(x^*)$ such that $y\notin L(x^*)$. Since $y\in\mathbf{U}(x^*)$ by transitivity of $\succsim$, we conclude that $\mathbf{U}(y)\subseteq\mathbf{U}(x^*)$. Consider $x^1 = \min\{y, x^*\}$. Suppose that $x^1 \neq y$. By **A.1** and our last conclusion, $\mathbf{U}(x^1) = \mathbf{U}(x^*)$ Hence, $x^1 \sim x^*$. Let $x^{\alpha_1} = \alpha_1 x^1 + (1-\alpha_1)x^*$, for $\alpha_1\in[0,1]$. Note that by **A.1**, $\mathbf{U}(x^*)\subseteq\mathbf{U}(x^{\alpha_1})\subseteq\mathbf{U}(x^1)$. But since $\mathbf{U}(x^1) = \mathbf{U}(x^*)$, we conclude that $\mathbf{U}(x^1) = \mathbf{U}(x^{\alpha_1}) = \mathbf{U}(x^*)$, which implies that $x^1 \sim x^{\alpha_1} \sim x^*$ for $\alpha_1\in[0,1]$. Let $x^2 = \pi_{1,2}(x^1)$ be the symmetric image of $x^1 : x^2 = (x^1_2, x^1_1)$. Then by **A.2**, $x^2 \sim x^{\alpha_2} \sim x^*$ where $x^{\alpha_2} = \alpha_2 x^2 + (1-\alpha_2)x^*$, for $\alpha_2\in[0,1]$. For any $\alpha_1,\alpha_2\in[0,1]$, let $x^{\alpha_1,\alpha_2} = \min\{x^{\alpha_1}, x^{\alpha_2}\}$. Then by **A.1**, $x^{\alpha_1,\alpha_2} \sim x^{\alpha_1} \sim x^{\alpha_2} \sim x^*$, which implies that alternatives in a square with vertices at $\{x^1, x^*, x^2, \min\{x^1, x^2\}\}$ (see Fig. 1) are indifferent to each other. But that contradicts **A.3**.

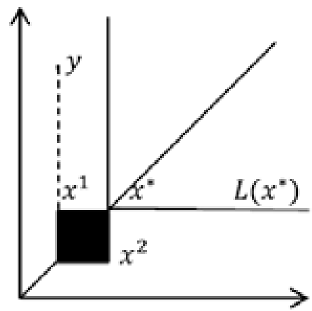


Figure 1. $x^1 \neq y$

Let's consider the other case. Suppose $x^1 = y$. Then by repeating the same argument we conclude that, alternatives in a quadrilateral with vertices at $\{y, x^*, y', \min\{y, y'\}\}$ where $y' = (y_2, y_1)$ is the symmetric image of $y$ (see Fig. 2) are indifferent to each other, which contradicts **A.3**. Hence, we conclude that $\nexists\, y\in\mathbf{U}(x^*)$ such that $y\notin L(x^*)$. So, $\mathbf{U}(x^*)\subseteq L(x^*)$ and the claim is established.

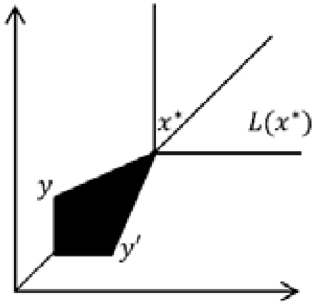


Figure 2. $x^1 = y$

**Step 2:** Let $\partial(L(x^*)) = \{x \in \mathbf{R}^2_+ : x_i \geq x^*_i, i = 1, 2, x_j = x^*_j$ for some $j = 1, 2\}$. We claim that $\mathbf{I}(x^*) = \partial(L(x^*))$. Take any $x \in \partial(L(x^*))$. Note that by **A.1**, $\mathbf{U}(x) \subseteq \mathbf{U}(x^*)$ since $\min\{x, x^*\} = x^*$. Hence, $x \succsim x^*$. Suppose $\exists x \in \partial(L(x^*))$ such that $x \succ x^*$. Consider $x' \in \partial(L(x^*))$ which is the symmetric image of $x$. Then, by **A.1**, $\mathbf{U}(x^*) = \mathbf{U}(x) \cup \mathbf{U}(x')$, since $x^* = \min\{x, x'\}$. By Step **1**, then $L(x^*) = \mathbf{U}(x) \cup \mathbf{U}(x')$. Since $x^* \notin \mathbf{U}(x)$, the last equality implies that $x^* \in \mathbf{U}(x')$. But since $x' \in \mathbf{U}(x^*) = L(x^*)$, we conclude that $x' \sim x^*$. But that contradicts **A.2**. So $\nexists$ $x \in \partial(L(x^*))$ such that $x \succ x^*$ and we conclude that $\forall x \in \partial(L(x^*)), x^* \succsim x$. Hence, $\partial(L(x^*)) \subseteq \mathbf{I}(x^*)$. For the other inclusion, suppose $\exists x \in \mathbf{I}(x^*)$ such that $x \notin \partial(L(x^*))$. But since by definition, $\mathbf{I}(x^*) \subseteq \mathbf{U}(x^*)$ and by Step **1**, $\mathbf{U}(x^*) = L(x^*)$, it must be the case that $x \in L(x^*)$. Hence, $\min\{x^*, x\} = x^*$. Let $x' \in \partial(L(x^*))$ be such that $x_i = x'_i$ for some $i = 1, 2$, i.e. projection of $x$ into $\partial(L(x^*))$. Note that $x \sim x^* \sim x'$ since the first conclusion is by definition and the second is by the statement just shown, $\partial(L(x^*)) \subseteq \mathbf{I}(x^*)$, which implies that $\mathbf{U}(x) = \mathbf{U}(x') = \mathbf{U}(x^*)$. Consider $x^{\alpha'} = \alpha' x + (1-\alpha')x'$ and $x^{\alpha^*} = \alpha^* x + (1-\alpha^*)x^*$ for $\alpha', \alpha^* \in [0,1]$. Note that by **A.1**, $\mathbf{U}(x) \subseteq \mathbf{U}(x^{\alpha'}) \subseteq \mathbf{U}(x')$ and $\mathbf{U}(x) \subseteq \mathbf{U}(x^{\alpha^*}) \subseteq \mathbf{U}(x^*)$, which implies that $\mathbf{U}(x) = \mathbf{U}(x^{\alpha'}) = \mathbf{U}(x') = \mathbf{U}(x^{\alpha^*}) = \mathbf{U}(x^*)$, hence $x \sim x^{\alpha'} \sim x' \sim x^{\alpha^*} \sim x^*$ for $\alpha', \alpha^* \in [0,1]$. For $\alpha', \alpha^* \in [0,1]$, let $x^{\alpha',\alpha^*} = \min\{x^{\alpha'}, x^{\alpha^*}\}$. Then by **A.1**, $\forall \alpha', \alpha^* \in [0,1]$, $x^{\alpha',\alpha^*} \sim x^{\alpha'} \sim x^{\alpha^*} \sim x^*$, which implies that alternatives in a triangle with vertices at $\{x, x', x^*\}$ (see Fig. 3) are indifferent to each other. But that contradicts **A.3**. Hence, we conclude that $\nexists$ $x \in \mathbf{I}(x^*)$ such that $x \notin \partial(L(x^*))$ and $\mathbf{I}(x^*) \subseteq \partial(L(x^*))$.

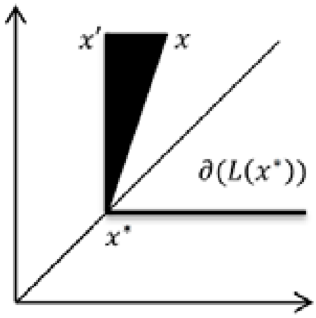


Figure 3. $x \in \mathbf{I}(x^*), x \notin \partial(L(x^*))$

**Step 3:** Suppose $y, z \in \mathbf{R}^2_+$ are such that $y \succsim z$ and let $y^*, z^* \in \mathbf{R}^2_+$ be such that

$$y^* = (\min\{y_1, y_2\}, \min\{y_1, y_2\})$$

and

$$z^* = (\min\{z_1, z_2\}, \min\{z_1, z_2\}).$$

Then, by construction $y \in \partial(L(y^*))$ and $z \in \partial(L(z^*))$. By Step **2**, $y \sim y^*$ and $z \sim z^*$ which implies $y^* \succsim z^*$. By Step **1**, $y^* \succsim z^*$ $\Leftrightarrow \min\{y_1, y_2\} \geq \min\{z_1, z_2\}$ and hence, $\succsim$ is the Leontief preferences.

(**b**) Starting with a point $x^* \in l(a_1, a_2)$ and repeating the same arguments as above one can show that (**1**) $\mathbf{U}(x^*) = L(x^*)$ and (**2**) $\mathbf{I}(x^*) = \partial(L(x^*))$. Suppose $y, z \in \mathbf{R}^2_+$ are such that $y \succsim z$ and let $y^*, z^* \in l(a_1, a_2)$ be such that

$$y^* = (\frac{1}{a_1}\min\{a_1 y_1, a_2 y_2\}, \frac{1}{a_2}\min\{a_1 y_1, a_2 y_2\})$$

and

$$z^* = (\frac{1}{a_1}\min\{a_1 z_1, a_2 z_2\}, \frac{1}{a_2}\min\{a_1 z_1, a_2 z_2\}).$$

Then, by construction $y \in \partial(L(y^*))$ and $z \in \partial(L(z^*))$. By (**2**), $y \sim y^*$, $z \sim z^*$ which implies that $y^* \succsim z^*$. By (**1**), $y^* \succsim z^*$ $\Leftrightarrow \min\{a_1 y_1, a_2 y_2\} \geq \min\{a_1 z_1, a_2 z_2\}$ and hence,

$\gtrsim$ is an $(a_1, a_2)$-weighted Leontief preferences. □

There are two commonly accepted criterion for the validity of an axiomatization result: consistency and logical independence (see also discussions in Chap. 1 in Kreps (1988)). Consistency of **A.1**, **A.2** (or **A.2'**) and **A.3** is established by the IF part of the characterization theorem above, and their independence can easily be verified:

**Example 1.** A preference satisfying **A.1** and **A.2** but not **A.3** is as follows (the arrow indicates the direction of utility increase):

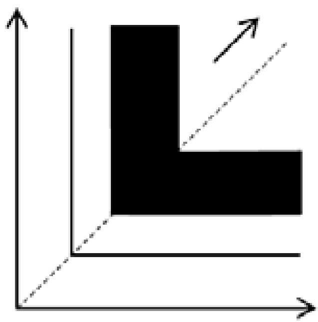

Figure 4. **A.3** is not satisfied

**Example 2.** A preference satisfying **A.1** and **A.3** but not **A.2** is as follows:

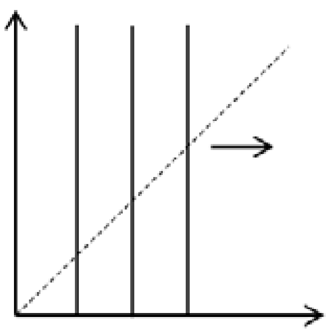

Figure 5. **A.2** is not satisfied

**Example 3.** A preference satisfying **A.2** and **A.3** but not **A.1** is as follows:

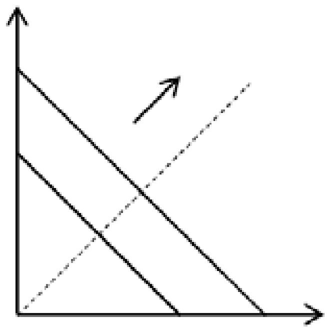

Figure 6. **A.1** is not satisfied

## *4. Conclusions*

This paper axiomatizes weak preferences on $\mathbf{R}_+^n$ as the Leontief preferences with three basic axioms: upper consistency, neutrality (or symmetry) and local non-satiation. Among the axioms, local non-satiation is standard in economic literature while neutrality (or symmetry) is also used, especially in the context of social choice (see for instance, *anonymity* in Lauwers (1997); and *symmetry* in Segal and Sobel (2002)). However, the upper consistency axiom is, to our best knowledge, new to the field.

Then one could ask whether upper consistency is related to the other axioms, especially to those used in the axiomatization of the *maximin* social welfare ordering, mentioned above. In this respect, it can be shown that upper consistency and local non-satiation together imply monotonicity (or weak Pareto) (see Lemma 2 in Appendix A). Also, an example of preferences that satisfy upper consistency, but not Hammond equity can be given (see Example 2 in Section 3), and it can easily be checked that the following preferences satisfy Hammond equity but not upper consistency:

$$\forall x \in \mathbf{R}_+^2, u(x) = -\max\{x_1, x_2\}.$$

## *Appendix A.*

For $x, y \in \mathbf{R}^n_+$, we write $x >> y$ if $x_i > y_i$ for all $i = 1, \ldots, n$. We say that $\succsim$ on $\mathbf{R}^n_+$ is

**A.4:** *Monotonic* if whenever $x, y \in \mathbf{R}^n_+$ are such that $x >> y$, we have $x \succ y$.

**Lemma 2.** If $\succsim$ on $\mathbf{R}^n_+$ satisfies **A.1** and **A.3**, then it satisfies **A.4**.

**Proof.** Consider $x, y \in \mathbf{R}^n_+$ such that $x >> y$. Then, $\min\{x, y\} = y$ and by **A.1**, $\mathbf{U}(x) \subseteq \mathbf{U}(y)$, which implies that $x \succsim y$. Suppose $x \sim y$ and consider the following $n$ -dimensional box $\mathbf{B}(x, y) = \{z \in \mathbf{R}^n_+ : x_i \geq z_i \geq y_i, i = 1, \ldots, n\}$. Then, for any $z \in \mathbf{B}(x, y), \min\{x, z\} = z$ and $\min\{z, y\} = y$ and by **A.1**, $x \succsim z$ and $z \succsim y$. Since $x \sim y$, we then conclude that $x \sim z \sim y$. But that contradicts **A.3**. Hence, $x \succ y$. □

**Theorem 3.** Let $\succsim$ be a weak order on $\mathbf{R}^n_+$. Then,

**(a)** $\succsim$ *satisfies A.1, A.2 and A.3 if and only if it is the Leontief preferences, and*

**(b)** $\succsim$ *satisfies A.1, A.2' and A.3 if and only if it is an* ***a****-weighted Leontief preferences.*

**Proof.** Since IF parts are easy to check, we prove ONLY IF parts. By Lemma 2 we may assume that $\succsim$ satisfies **A.4**.

**(a)** Suppose $\succsim$ satisfies **A.1–A.4**. Consider $x^* \in l$. Let

$$L(x^*) = \{x \in \mathbf{R}^n_+ : x_i \geq x^*_i, i = 1, \ldots, n\}.$$

We proceed in 3 steps.

**Step 1(a):** We claim that $\mathbf{U}(x^*) = L(x^*)$. First, note that for any $x \in L(x^*), \min\{x, x^*\} = x^*$. Then by **A.1**, $\mathbf{U}(x) \subseteq \mathbf{U}(x^*)$. In particular, $x \in \mathbf{U}(x^*)$. Hence, $L(x^*) \subseteq \mathbf{U}(x^*)$. For the other inclusion, suppose $\exists y \in \mathbf{U}(x^*)$ such that $y \notin L(x^*)$. Since $y \in \mathbf{U}(x^*)$, by transitivity of $\succsim$, we conclude that $\mathbf{U}(y) \subseteq \mathbf{U}(x^*)$. Consider $x = \min\{y, x^*\}$. By **A.1** and by our last conclusion, $\mathbf{U}(x) = \mathbf{U}(x^*)$ and hence, $x \sim x^*$. Since $y \notin L(x^*)$, it is the case that $x \notin L(x^*)$ and hence $\exists j \in \{1, \ldots, n\}$ such that $x_j < x^*_j$. For $i = 1, \ldots, n$, let $x^i = \pi_{i,j}(x)$. Then by **A.2**, $x^i \sim x^*$, $i = 1, \ldots, n$. Let

$$x^{\min} = \min\{x^n, \min\{x^{n-1}, \min\{x^{n-2}, \ldots, \min\{x^3, \min\{x^2, x^1\}\} \ldots\}\}\}.$$

Then by repeated use of **A.1** and by our conclusion that $x^i \sim x^*$, $i = 1, \ldots, n$, we conclude that $x^{\min} \sim x^*$. Note that by construction, $x^{\min}_i \leq x_j, i = 1, \ldots, n$ which implies that $x^{\min} < x^*$. But that contradicts **A.4**. Hence, we conclude that $\mathbf{U}(x) \subseteq L(x^*)$.

**Step 2(a):** Let

$\partial(L(x^*)) = \{x \in \mathbf{R}^n_+ : x_i \geq x^*_i, i = 1, \ldots, n, x_j = x^*_j \text{ for some } j = 1, \ldots, n\}$.

We claim that $\mathbf{I}(x^*) = \partial(L(x^*))$. Since by definition, $\mathbf{I}(x^*) \subseteq \mathbf{U}(x^*)$, by Step **1(a)** we conclude that $\mathbf{I}(x^*) \subseteq L(x^*)$. Then by **A.4**, $\mathbf{I}(x^*) \subseteq \partial(L(x^*))$. For the other inclusion, consider any $x \in \partial(L(x^*))$. Then by Step **1(a)**, $x \succsim x^*$. Suppose $\exists x \in \partial(L(x^*))$ such that $x \succ x^*$. Since $x \in \partial(L(x^*))$, $\exists j \in \{1, \ldots, n\}$ such that $x_j = x^*_j$. For $i = 1, \ldots, n$, let $x^i = \pi_{i,j}(x)$. Let

$$x^{\min} = \min\{x^n, \min\{x^{n-1}, \min\{x^{n-2}, \ldots, \min\{x^3, \min\{x^2, x^1\}\} \ldots\}\}\}.$$

Note that by construction, $x^{\min} = x^*$. Then, by repeated use of **A.1** we conclude that $\mathbf{U}(x^*) = \mathbf{U}(x^1) \cup \ldots \cup \mathbf{U}(x^n)$. Hence, $\exists k \in \{1, \ldots, n\}$ such that $x^* \in \mathbf{U}(x^k)$, which implies that $x^* \sim x^k$. Then by **A.2**, $x^* \sim x^j = x$, which is a contradiction. Hence, $\partial(L(x^*)) \subseteq \mathbf{I}(x^*)$.

**Step 3(a):** Suppose $y, z \in \mathbf{R}^n_+$ are such that $y \succsim z$ and let $y^*, z^* \in l$ be such that

$$y^* = (\min\{y_1, \ldots, y_n\}, \ldots, \min\{y_1, \ldots, y_n\})$$

and

$$z^* = (\min\{z_1, \ldots, z_n\}, \ldots, \min\{z_1, \ldots, z_n\}).$$

Then, by construction $y \in \partial(L(y^*))$ and $z \in \partial(L(z^*))$. By Step **2(a)**, $y \sim y^*$ and $z \sim z^*$ which implies that $y^* \gtrsim z^*$. Then, by Step **1(a)**,

$$y^* \gtrsim z^* \Leftrightarrow \min\{y_1,\ldots,y_n\} \geq \min\{z_1,\ldots,z_n\}$$

and hence, $\gtrsim$ is the Leontief preferences.

**(b)** Consider $x^* \in l(a_1,\ldots,a_n)$. Note that $\forall i, j = 1,\ldots,n,$

$$(\frac{1}{a_1},\ldots,\frac{1}{a_n}) * \pi_{i,j}(a_1x_1^*,\ldots,a_nx_n^*) = x^*$$

i.e. the neutral (symmetric) image of $x^*$ is itself, since $a_ix_i^* = a_jx_j^* \Leftrightarrow (1/a_i)a_jx_j^* = x_i^*$. We proceed in 3 steps.

**Step 1(b):** We claim that $\mathbf{U}(x^*) = L(x^*)$. First, by repeating the same argument as in Step **1(a)** we conclude that $L(x^*) \subseteq \mathbf{U}(x^*)$. For the other inclusion, suppose $\exists y \in \mathbf{U}(x^*)$ such that $y \notin L(x^*)$. Since $y \in \mathbf{U}(x^*)$, by transitivity of $\gtrsim$, we conclude that $\mathbf{U}(y) \subseteq \mathbf{U}(x^*)$. Consider $x = \min\{y, x^*\}$. By **A.1** and by our last conclusion, $\mathbf{U}(x) = \mathbf{U}(x^*)$ and hence, $x \sim x^*$. Since $y \notin L(x^*)$, $x \notin L(x^*)$ and $\exists j \in \{1,\ldots,n\}$ such that $x_j < x_j^*$.

For $i = 1,\ldots,n,$ let $x^i = (1/a_1,\ldots,1/a_n) * \pi_{i,j}(a_1x_1,\ldots,a_nx_n)$. Then by **A.2'**, $x^i \sim x^*$, $i = 1,\ldots,n$. Let

$$x^{\min} = \min\{x^n, \min\{x^{n-1}, \min\{x^{n-2},\ldots,\min\{x^3,\min\{x^2,x^1\}\}\ldots\}\}\}.$$

Then by repeated use of **A.1** and by our conclusion that $x^i \sim x^*$, $i = 1,\ldots,n,$ we conclude that $x^{\min} \sim x^*$. Note that by construction, the *i'th* component of $x^i$ is $x_i^i = (1/a_i)a_jx_j$, for $i = 1,\ldots,n.$ Then, $x_i^i < x_i^*$, for $i = 1,\ldots,n$ since $(1/a_i)a_jx_j < (1/a_i)a_jx_j^* = (1/a_i)a_ix_i^* = x_i^*$, which implies that $x^{\min} << x^*$. But that contradicts **A.4**. Hence, we conclude that $\mathbf{U}(x^*) \subseteq L(x^*)$.

**Step 2(b):** We claim that $\mathbf{I}(x^*) = \partial(L(x^*))$. First, by repeating the same argument as in Step **2(a)** we conclude that $\mathbf{I}(x^*) \subseteq L(x^*)$. Then by **A.4**, $\mathbf{I}(x^*) \subseteq \partial(L(x^*))$. For the other inclusion, consider any $x \in \partial(L(x^*))$. Then by Step **1(b)**, $x \gtrsim x^*$. Suppose $\exists x \in \partial(L(x^*))$ such that $x \succ x^*$. Since $x \in \partial(L(x^*))$, $\exists j \in \{1,\ldots,n\}$ such that $x_j = x_j^*$. For $i = 1,\ldots,n,$ let $x^i = (1/a_1,\ldots,1/a_n) * \pi_{i,j}(a_1x_1,\ldots,a_nx_n)$. Let

$$x^{\min} = \min\{x^n, \min\{x^{n-1}, \min\{x^{n-2},\ldots,\min\{x^3,\min\{x^2,x^1\}\}\ldots\}\}\}.$$

Note that by construction, $\forall i = 1,\ldots,n,$ the *i'th* component of $x^i$ is $x_i^i = (1/a_i)a_jx_j = (1/a_i)a_jx_j^* = (1/a_i)a_ix_i^* = x_i^*$, and when $i \neq j$, $\forall k \in \{1,\ldots,n\}\backslash\{i\}$, the *i´th* component of $x^k$ is $x_i^k = x_i \geq x_i^*$ (recall that $x \in \partial(L(x^*))$), and when $i = j$, $\forall k \in \{1,\ldots,n\}\backslash\{i\}$, the *j'th* component of $x^k$ is $x_j^k = (1/a_j)a_kx_k \geq (1/a_j)a_kx_k^* = (1/a_j)a_jx_j^* = x_j^*$. This implies that $x^{\min} = x^*$. Then, by repeated use of **A.1** we conclude that

$$\mathbf{U}(x^*) = \mathbf{U}(x^1) \cup \ldots \cup \mathbf{U}(x^n).$$

Hence, $\exists q \in \{1,\ldots,n\}$ such that $x^* \in \mathbf{U}(x^q)$, which implies that $x^* \sim x^q$. Then by **A.2'**, $x^* \sim x^j = x$, which is a contradiction. Hence, $\partial(L(x^*)) \subseteq \mathbf{I}(x^*)$.

**Step 3(b):** Suppose $y, z \in \mathbf{R}_+^n$ are such that $y \gtrsim z$ and let $y^*, z^* \in l(a_1,\ldots,a_n)$ be such that

$$y^* = (\frac{1}{a_1}\min\{a_1y_1,\ldots,a_ny_n\},\ldots,\frac{1}{a_n}\min\{a_1y_1,\ldots,a_ny_n\})$$

and

$$z^* = \left(\frac{1}{a_1}\min\{a_1 z_1,\ldots,a_n z_n\},\ldots,\frac{1}{a_n}\min\{a_1 z_1,\ldots,a_n z_n\}\right).$$

Then, by construction $y \in \partial(L(y^*))$ and $z \in \partial(L(z^*))$. By Step **2(b)**, $y \sim y^*$ and $z \sim z^*$ which implies that $y^* \gtrsim z^*$. Then, by Step **1(b)**,

$$y^* \gtrsim z^* \Leftrightarrow \min\{a_1 y_1,\ldots,a_n y_n\} \geq \min\{a_1 z_1,\ldots,a_n z_n\}$$

and hence, $\gtrsim$ is an **a**-weighted Leontief preferences. □

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