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REVISITING THE NORMALIZATION AXIOM IN POVERTY MEASUREMENT*

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The 'normalization' axiom associated with Sen's poverty index – and this, indeed, holds for most extant measures of poverty – entails an uncomfortable implication when we adopt a strong, or inclusive, definition of the poor. This paper suggests that we may not always be at liberty to adopt a weak definition. The available alternative then is to change the form of the poverty measure. Accordingly, a modification of his normalization axiom which leads to a variant of Sen's index, together with a variant also of the Foster-Greer-Thorbecke poverty measures, is advanced and discussed. The derivation of the new normalization axiom benefits from Basu's decomposition of the Sen axiom.

1. Introduction: Classifying the Population by Poverty Status

The present paper could be seen as a specific example of a difficulty that can arise from a 'strong' (or 'inclusive') classification of the relevant population (poor or non-poor) by poverty status. It sometimes happens that what appears to be a minor or innocuous difference – such as whether to include or exclude those on the poverty line from one's definition of the poor or the non-poor – can actually have rather material consequences for the outcomes of measurement. To quote Donaldson and Weymark (1986; pp. 668, 670, 687):

> We consider two definitions of the poor. In the weak definition, the poor consists of all individuals with incomes strictly less than the poverty line. The strong definition of the poor expands the group of poor people to include anyone with an income equal to the poverty line as well. With the weak definition of the poor, all of the axioms we consider are compatible. However, with the strong definition of the poor, a number of impossibilities occur … This seemingly trivial difference in definitions of the poor has important implications … [A]n issue that might seem to be of minor importance, namely the classification of individuals at the poverty line, has significant implications.

This paper explores one such implication, in the context of an axiom called the *normalization axiom* (to be hereafter referred to as Axiom N) employed by Sen (1976, 1979) in the derivation of his real-valued index of poverty. The nor-

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malization axiom states the following. Let H be the *headcount ratio* (or proportion of the population in poverty), and I the *income-gap ratio* (or proportionate shortfall of the average income of the poor from the poverty line). Then, Axiom N demands that, in the special case in which all the poor have the same income, the poverty index should be equal to the *product* of the headcount ratio and the income-gap ratio. If the poor are defined to include all individuals with incomes not exceeding a threshold 'poverty line' level of income, then this multiplicative combination of H and I entailed by Axiom N can be problematic, in a specific sense, which is explicated in what follows.

It should be clarified, at the very outset, that whether or not Axiom N could be 'problematic' would depend on how one chooses to define the populations of the poor and the non-poor. There is one scheme of classification (the 'exclusive' scheme, to be discussed later) under which the issue simply 'dissolves', and another scheme (the 'inclusive' scheme) under which there is a genuine difficulty that needs to be addressed. It must be stated clearly that for those who are persuaded of the unique rightness of the first of these classificatory schemes, this paper must appear to deal with a non-issue. The concerns of the paper, that is, would make any sense only if it is conceded that the alternative (inclusive) classificatory scheme cannot be straightforwardly rejected as unreasonable. To see what is involved, it is useful to consider some preliminary definitions of terms and concepts.

Let me motivate the discussion in a slightly oblique fashion, by recalling the work of Watts (1968), who proposed a unified approach to the measurement of poverty, of 'affluence' (or, perhaps more appropriately, 'non-poorness') , and of a welfare-related combination of both. I focus specifically on Watts because he was the first writer to advance a conjoint treatment of the measurement of poverty, of affluence, and of welfare – by seeing the affluence index as a reflected version of the poverty index, and the welfare index as a simple sum of the poverty and affluence indices. The specificity with respect to Watts, however, need not obscure a certain wider and more general domain of applicability of the concerns under review. In principle, given any measure of poverty, one could conceive of a measure of non-poorness as a sort of mirror-image of the poverty measure. This – with appropriate modifications, of course - has indeed been advocated, with respect both to the Sen index of poverty (on which see P. K. Sen, 1988), and the Foster-Greer-Thorbecke class of poverty measures (on which see Peichi, Schaefer and Scheicher, 2008). The central issue involved here is that, if it is legitimate to measure both poverty and non-poorness, then either the poor or the non-poor population may have to be classified in terms of an inclusive definition. This is true, whether or not any particular poverty index was conceived of in conjunction with a complementary 'non-poorness' index, and so applies, *inter alia*, to the Sen index and the Foster-Greer-Thorbecke indices, although the motivating discussion, which follows, will be conducted with reference to the Watts formulation.

Let $\{1,\ldots,i,\ldots,n\} \equiv N$ be the set of individuals constituting society, let $x_i (i = 1,\ldots,n)$ be the income of person i, and let z be the threshold, 'poverty line' level of income. Define the following three mutually exclusive and completely exhaustive subsets of N:

$$S_1 \equiv \{i \in N | x_i < z\};$$

$$S_2 \equiv \{i \in N | x_i = z\}; \text{ and}$$

$$S_3 \equiv \{i \in N | x_i > z\}.$$

The Watts measures of poverty and 'non-poorness' (or 'affluence', as Watts (1968) called it) could be seen to be given, respectively, by $W^P = (1/n)\sum_{i \in S_1} \log_e(z/x_i)$ if S_1 is non-empty and zero otherwise, and $W^{NP} = (1/n)\sum_{i \in S_3} \log_e(z/x_i)$ if S_3 is non-empty and zero otherwise. (Watts also went on to define an overall measure of welfare W^W given by summing the measures of poverty and of affluence $[W^W = W^P + W^{NP}]$[1]: this is men-

[1] *If $W^P > |W^{NP}|$, then W^W will be positive. By normal convention, one would be inclined to treat welfare as positive when the extent of non-poorness exceeds the extent of poverty: to accommodate this, we could take the aggregate welfare measure to be given by $(-)W^W$ rather than by W^W.*

tioned here only for purposes of completeness, and will have no further role to play in this paper.) There is nothing inherently problematic about partitioning the population in this manner, and – as will become apparent shortly – the problem dealt with in this paper would not arise with the three-fold classificatory scheme (consisting of the sets S_1, S_2 and S_3) just defined. The unexceptionableness of this classificatory scheme comes through particularly strongly at a certain level of abstraction. However, suppose we were to take an 'ordinary language' view of the matter, and to adopt some common-sense convention of 'naming' the sets S_1, S_2 and S_3, so as to render the 'meanings' of the indices W^P and W^{NP} transparent. Then, it appears both reasonable and likely that S_1 would be defined as the set of the poor and S_3 as the set of the non-poor[2] – in which case, S_2 would have be defined as a set whose typical member is an individual *i* of whom it is true that s/he is not poor and also not non-poor. This falls foul of a semantic principle, the *Principle of Contradiction* (POC), which is a dual of another semantic principle, the *Bivalence Principle*. The former Principle demands of any proposition *p* that it cannot be such that both *p* and (not-*p*) are true. If the present scheme of classification by poverty status is seen to be unsatisfactory for the reason just discussed, then one would have to resort to an alternative partitioning of the population – one which can, as it turns out, have adverse implications for Sen's normalization axiom.

Under this alternative classificatory system, the population would have to be divided into *two* mutually exclusive and completely exhaustive subsets – those of the poor and the non-poor. Let Q be the set of poor individuals in N, and $\hat{Q}$ the set of non-poor individuals. ('Affluent' is a convenient, if not quite accurate, substitute for the more apposite term 'non-poor': the two are sometimes used interchangeably, but it does less violence to the use of language to employ the term 'non-poor', when that is what is meant.) Clearly, the set $\hat{Q}$ is the complement of the set Q in the set N, and the individuals belonging to Q and $\hat{Q}$ respectively are separated by the poverty line z. But separated how? If the non-poor individuals are defined 'inclusively', to signify those with incomes not less than z (that is, $\hat{Q}$ is defined such that $\hat{Q} \equiv \{i \in N | x_i \geq z\}$), then Q must necessarily be defined 'exclusively' (that is, such that $Q \equiv \{i \in N | x_i < z\}$); on the other hand, if $\hat{Q}$ is defined 'exclusively' (that is, such that $Q \equiv \{i \in N | x_i > z\}$), then Q must necessarily be defined 'inclusively' (that is, such that $Q \equiv \{i \in N | x_i \leq z\}$). The Watts measures of poverty and non-poorness would now be given, respectively, by $W^P = (1/n)\sum_{i \in Q} \log_e(z/x_i)$ if Q is non-empty and zero otherwise, and $W^{NP} = (1/n)\sum_{i \in \hat{Q}} \log_e(z/x_i)$ if $\hat{Q}$ is non-empty and zero otherwise.

Suppose $\hat{Q}$ to have been defined inclusively. Imagine two situations, labeled Situation 1 and Situation 2 respectively. In Situation 1 there is exactly one individual with an income of z, while each of the remaining (n–1) individuals has an income of z/2 each. In Situation 2, all n individuals have an income of z each. It is easy to see that, in Situation 1, $W^P = [(n-1)/n]\log_e 2$, while in Situation 2, $W^P = 0$: the poverty level in Situation 1 is finite, while there is no poverty in Situation 2, which seems to be a reasonable enough judgment, considering that a finite number of persons in Situation 1 are poor while no-one in Situation 2 is poor. Next, consider the non-poorness measure W^{NP}. It is easy to see that $W^{NP} = 0$ in both Situations 1 and 2: this must certainly be judged odd, since there is only one non-poor person (with an income of z) in Situation 1, while all n persons out of n, with the same income of z each, are non-poor in Situation 2.

The oddness of this judgment can, of course, be easily rectified, by switching from the inclusive to the exclusive definition of the non-poor: in this case, the set $\hat{Q}$ would be empty in both Situations 1 and 2, and W^{NP} would – not unreasonably – be zero in both Situations. The difficulty with this 'easy rectification', however, is that with $\hat{Q}$ now being defined exclusively, Q

[2] *For instance, in targeted poverty alleviation schemes, people on the poverty line must either be included within or excluded from the ambit of programme benefits: if they are included, it is presumably because they are regarded as being poor, and if they are excluded, it is presumably because they are regarded as being non-poor.*

would necessarily have to be defined inclusively. Now imagine two situations, labeled Situation 3 and Situation 4 respectively, such that, in Situation 3, there is exactly one person with an income of z and there are $(n–1)$ persons with an income of $3z/2$ each, while in Situation 4 (as in Situation 2), each of the n persons has an income of z each. Under the inclusive definition of Q, it is easy to verify that $W^P = 0$ in each of Situations 3 and 4 – which is an odd judgment given that in Situation 3 there is only one person (with an income of z) who is poor, while all n persons out of n, with the same income of z each, are poor in Situation 4. Briefly, a form of resolution of the problem which entails switching from the inclusive to the exclusive definition is not very satisfactory, because it effectively tells an analyst that she may measure poverty or non-poorness, but not both, without allowing odd judgments to mediate the exercise. A more promising approach to resolution may, instead, require a re-consideration of the measures of poverty (or non-poorness) themselves.

Let me re-iterate, at this point, that the concerns of this paper will be seen to have any validity only if some merit is conceded to the notion that it may be problematic to define *both* the poor and the non-poor populations exclusively, for reasons discussed in the preceding paragraphs. If this concession is made, then one can examine – without loss of generality – the consequences for measurement of defining the poor population inclusively, as is done in what follows.

2. The Sen Normalization under an Inclusive Definition of the Poor

An asymptotic expression for Sen's index of poverty (that is, in a situation where the numbers of the poor are 'large') – see Sen (1976) - is given by

$$(2.1) \quad P^S = H[I + (1–I)G^P],$$

where G^P is the Gini coefficient of inequality in the distribution of income among the poor. Notice that *if the set of poor individuals is defined so as to include those with incomes equal to the poverty line*, then when I is zero because all the poor have the same income, equal to the poverty line, clearly G^P is zero, and hence – in view of (2.1) – P^S is zero. This is an undesirable property of the poverty index: for any given population and any given poverty line, any two equal distributions of income among the poor in which all the poor receive the poverty line income must be judged to have the same extent of poverty, despite the fact that only one person out of one hundred may be poor under the first distribution, while one hundred persons out of one hundred may be poor under the second. That is, Sen's poverty index does not allow for monotonicity in H when I is zero by virtue of all poor individuals sharing the poverty line income. This is ensured by the multiplicative form entailed by the normalization axiom, which demands that when there is no inequality in the distribution of poor incomes, poverty is given by the product of H and I.

Indeed, this difficulty is a feature also of the Foster-Greer-Thorbecke (1984) family of poverty indices, given by

$$(2.2) \quad P_\alpha = (1/n)\sum_{i=1}^{q}[(z-x_i)/z]^\alpha, \alpha \geq 0,$$

where z is the poverty line, x_i is the income of the ith person in the set of poor persons, q is the number of poor persons, n is the total population, and α is a measure of 'poverty aversion'. For given n and z, and for all $\alpha > 0$, it is easy to see from (2.2) that all distributions in which $x_i = z \forall i = 1,\ldots,q$, will be certified by P_α to reflect zero poverty, irrespective of the value of q in relation to n.

As we have noted earlier, it is, of course, true that a simple way out of the problem just discussed would be to exclude individuals *on* the poverty line from the definition of the poor. But – and this was also noted earlier – exclusive definitions of the poor or of the non-poor are, from one point of view, compatible with 'satisfactory' measurement of only either poverty or non-poorness, but not of both. This is not a very appealing way out of the problem, if it is conceded that it is, after all, legitimate to be interested in comparing distributions not only in terms of poverty but also in terms of 'non-poorness'.

Under these circumstances, there may be a case for taking another look at the normalization axiom itself. Sen's normalization, entailing a multiplicative combination of H and I is, as we have seen, not entirely satisfactory. Takayama (1979) proposed a normalization axiom in which – when poor incomes are equally distributed – the extent of poverty is identified with the *minimum* of the headcount ratio and the income-gap ratio. For a given population and a given poverty line, Takayama's normalization also fails, when all poor incomes coincide with the poverty line, to allow for monotonicity of the poverty measure in the headcount ratio. This is true also for the normalization axiom proposed by Pattanaik and Sengupta (1995), which demands that the poverty measure should (a) be zero when all poor incomes are equal to the poverty line, and (b) coincide with the headcount ratio when all poor incomes are equal to zero.

In this paper, I propose a normalization axiom (N*) in which the extent of poverty is equated to *one-half the sum* of the headcount ratio and the income-gap ratio. Axiom N* is sought to be rationalized within an axiomatic framework. In the process, some aspects of Basu's (1985) rationalization of Sen's normalization axiom N are reviewed, and the relevance and usefulness of Basu's arguments for justifying Axiom N* are noted. Axiom N* can be used (in conjunction with the other axioms proposed by Sen for justifying his own poverty index P^S) to derive a variant P^{S*} of P^S. P^{S*} is free of a particular difficulty which is a feature of P^S, namely the failure of the poverty index to be a monotonically increasing function of the headcount ratio, in the special case in which the strong definition of the poor is adopted and all the poor have an income equal to the poverty line. We have noted that this difficulty is also a feature of the Foster-Greer-Thorbecke (1984) P_α family of poverty indices. A minor change in the specification of the functional form of the poverty index will rectify this difficulty: a family of poverty indices P_α^* which are variants of the indices in the P_α family are presented in this paper.

The new normalization axiom, and the corresponding variants of Sen's index and the Foster-Greer-Thorbecke indices, however, are secured at a possible price, namely the violation of certain other canonically valued axioms of poverty measurement. These difficulties are discussed in the paper. Overall, the essay points to the troublesome possibility that seemingly rather small issues can result in rather large complications for the measurement of poverty.

3. Axiom N* and its Rationalization

I start with some basic definitions, drawing on Basu (1985). Let $\mathbf{X}^n \equiv \{\mathbf{x} \in \mathrm{R}_+^n \mid x_i \le x_{i+1}, i = 1,\ldots,n-1\}$, where R_+^n is the non-negative orthant of n-dimensional real space. Every $\mathbf{x}$ belonging to $\mathbf{X} \equiv \bigcup_{n=1}^{\infty} \mathbf{X}^n$ is then a description of an n-person ordered distribution of (non-negative) incomes. z is the *poverty line*, which is a level of income such that any individual whose income does not exceed z is certified to be poor. (This, to recall, is the 'strong' definition of the poor). For every $\mathbf{x} \in \mathbf{X}$, let $S(\mathbf{x})$ be the set of poor individuals in $\mathbf{x}$. A *poverty measure* is a function such that, for all $\mathbf{x} \in \mathbf{X}$, $0 \le P(\mathbf{x}) \le 1$. The *headcount ratio H* and the *income-gap ratio I* are defined as follows:

$$\forall \mathbf{x} \in \mathbf{X}, H(\mathbf{x}) = \# S(\mathbf{x}) / \# \mathbf{x}.$$

$$\forall \mathbf{x} \in \mathbf{X}, I(\mathbf{x}) = 0 \text{ if } S(\mathbf{x}) = \{\} \text{ and}$$

$$I(\mathbf{x}) = \sum_{i \in S(\mathbf{x})} (z - x_i) / (\# S(\mathbf{x}) \cdot z), \text{ otherwise.}$$

Let $\mathbf{X}^* (\subset \mathbf{X}) \equiv \{\mathbf{x} \in \mathbf{X} \mid x_i = x_{i-1} \forall i \in S(\mathbf{x}) \setminus \{1\}\}$. It will be assumed that there exists a function f such that

$$(3.1) \quad \forall \mathbf{x} \in \mathbf{X}^*, P(\mathbf{x}) = f(H(\mathbf{x}), I(\mathbf{x})).$$

It is useful, first, to consider an approach to rationalizing Sen's normalization axiom, of the type adopted by Basu (1985). Sen's normalization axiom can be stated as follows.

Axiom N. $\forall \mathbf{x} \in \mathbf{X}^*, P(\mathbf{x}) = f(H(\mathbf{x}), I(\mathbf{x})) = H(\mathbf{x}) \cdot I(\mathbf{x})$.

As pointed out by Basu, $H(\mathbf{X}^*) \equiv \mathrm{Q} \cap [0,1]$, where Q is the set of all rational numbers (note that since $\# S(\mathbf{x})$ and $\# \mathbf{x}$ are both integers, the headcount ratio must be a rational number), while $I(\mathbf{X}^*) \equiv [0,1]$. Given this, Basu takes the function f to be a mapping from $(Q\cap[0,1])x[0,1]$ to $[0,1]$. Properly speaking, however, the domain of the function f is a strict subset of $(Q\cap[0,1])x[0,1]$. For notice that when H is zero, I must also be zero: combinations of H and I for which H is zero and I is finite do not exist. Let $\{\mathbf{0}\}$ be the point (0,0) in 2-dimensional real space. Define the set $M \equiv \{\mathbf{0}\}\cup(Q\cap(0,1])x[0,1]$. Then, we can correctly assert that

(3.2) $f : M \rightarrow [0,1]$.

Basu now proposes the following axioms to justify Sen's normalization axiom.

Axiom 1(a). $f(1,1) = 1$.

Axiom 1(b). $Lim_{H\rightarrow 0}f(H,I) = Lim_{I\rightarrow 0}f(H,I) = 0$.

Axiom 2. $\forall H_1,H_2,H_3,H_4 \in Q\cap[0,1]$ and $\forall I \in [0,1]$, $[H_1 - H_2 > (=)\ H_3 - H_4] \rightarrow [f(H_1,I) - f(H_2,I) > (=)\ f(H_3,I) - f(H_4,I)]$.

Axiom 3. $\forall I_1,I_2,I_3,I_4 \in [0,1]$ and $\forall H \in Q\cap[0,1]$, $[I_1 - I_2 > (=)\ I_3 - I_4] \rightarrow [f(H,I_1) - f(H,I_2) > (=)\ f(H,I_3) - f(H,I_4)]$

Basu establishes that Axioms 1–3 are equivalent to Axiom N.

The preceding brief review paves the way for a set of axioms which uniquely imply the following variant of Sen's normalization axiom:

Axiom N*. $\forall \mathbf{x} \in \mathbf{X}^*$, $P(\mathbf{x}) = f(H(\mathbf{x}),I(\mathbf{x})) = [H(\mathbf{x}) + I(\mathbf{x})]/2$.

Towards rationalizing Axiom N*, I begin with a definition: every $\mathbf{m} = (m_1,m_2) \in M$ is a *poverty regime*, defined by an ordered pair of headcount ratio and income-gap ratio. (M, to recall, is the set $\{\mathbf{0}\}\cup(Q\cap(0,1])x[0,1]$. The following axioms are restrictions on the function $f : M \rightarrow [0,1]$.

Axiom I (Boundary Axiom)

(a) $f(0,0) = 0$;

(b) $f(1,1) = 1$.

That is, when there are no poor individuals, the extent of poverty is zero, while when everyone is poor and receives no income, the extent of poverty is unity.

Axiom II (Proportional Monotonicity Axiom)

(a) $\forall \mathbf{m},\mathbf{m}' \in M$ such that $m_2 = m'_2$, $f(\mathbf{m}) - f(\mathbf{m}') = k_1(m_1 - m'_1)$, where $k_1 \in R_{++}$;

(b) $\forall \mathbf{m},\mathbf{m}' \in M$ such that $m_1 = m'_1$, $f(\mathbf{m}) - f(\mathbf{m}') = k_2(m_2 - m'_2)$, where $k_2 \in R_{++}$.

That is, if two poverty regimes differ from each other only with respect to the headcount ratio (respectively, income-gap ratio), then the difference in the extent of poverty in the two regimes varies directly with the difference in the headcount ratios (respectively, income-gap ratios) in the two regimes. Notice that proportional monotonicity implies a stronger condition than monotonicity which would require only that, other things equal, a regime with a higher headcount ratio (respectively, income-gap ratio) should reflect greater poverty.

Axiom III (Symmetry Axiom) $\forall \mathbf{m},\mathbf{m}' \in M$ such that $m_1 = m'_2$, and $m_2 = m'_1$, $f(\mathbf{m}) = f(\mathbf{m}')$.

That is, the extent of poverty remains unchanged when one poverty regime is derived from another by a simple interchange of the headcount ratio and the income-gap ratio. (Loosely, the headcount ratio and the income-gap ratio are of 'equal importance' in determining the extent of poverty.) It must be admitted that the judgment embodied in Axiom III *is* somewhat arbitrary. For example, consider the poverty regimes (.1,1) and (1, .1): in the first regime, ten per cent of the population are in extreme deprivation while in the second regime, the burden of deprivation is more evenly spread among the entire population; it could well be held that a small incidence of acute deprivation is worse than a large incidence of relatively mild deprivation. For the purposes of this note, however, we shall stick with Axiom III; and it might be noted that Axiom N also satisfies the Symmetry Axiom.

The following proposition is now true.

Theorem 1. Let $f : M \rightarrow [0,1]$. Then, $\forall \mathbf{m} \in M$, $f(\mathbf{m}) = (m_1 + m_2)/2$ if and only if f satisfies Axioms I, II and III.

Proof (a). *Sufficiency*: Straightforward. (b) *Necessity*: Consider the poverty regime $(m_1,0) \in M$. By Axiom II(a),
$f(m_1,0) - f(0,0) = k_1 m_1$, $k_1 > 0$, or

(3.3) $f(m_1,0) = k_1 m_1$,

since $f(0,0) = 0$ by Axiom I(a).

Consider the regime $(m_1,m_2) \in M$. By Axiom II(b), $f(m_1,m_2) - f(m_1,0) = k_2 m_2$, $k_2 > 0$, whence

(3.4) $f(m_1,m_2) = k_1 m_1 + k_2 m_2$,

since $f(m_1,0) = k_1 m_1$ by (3.3).

Setting $m_1 = m_2 = 1$ in (3.4), and invoking Axiom 1(b), we have:

(3.5) $k_1 + k_2 = 1$.

Let (s,t) and (t,s) be two poverty regimes, with $s \neq t$. By virtue of (3.4), we have:

(3.6) $f(s,t) = k_1 s + k_2 t$;

and

(3.7) $f(t,s) = k_1 t + k_2 s$.

By Axiom III, $f(s,t) = f(t,s)$. This, in conjunction with (3.6) and (3.7), and recalling that $s \neq t$ by assumption, leads to

(3.8) $k_1 - k_2 = 0$.

From (3.5) and (3.8), we obtain:

(3.9) $k_1 = k_2 = 1/2$.

Substituting for k_1 and k_2 from (3.9) into (3.4) yields:
$\forall \mathbf{m} \in M$, $f(\mathbf{m}) = (m_1 + m_2)/2$, as desired. (*Q.E.D.*)

An alternative justification of the normalization axiom N* can be obtained by retaining Axioms I and III and replacing Axiom II by the following axiom proposed (and discussed) in Basu (1985):

Axiom II' (Difference Preservation Axiom).

(a) $\forall \mathbf{m}^1,\mathbf{m}^2,\mathbf{m}^3,\mathbf{m}^4 \in M$ such that $m_1^1 = m_1^2 = m_1^3 = m_1^4$,
$[(m_2^4 - m_2^3) > (=) (m_2^2 - m_2^1)] \rightarrow [f(\mathbf{m}^4) - f(\mathbf{m}^3) > (=) f(\mathbf{m}^2) - f(\mathbf{m}^1)]$.

(b) $\forall \mathbf{m}^1,\mathbf{m}^2,\mathbf{m}^3,\mathbf{m}^4 \in M$ such that $m_2^1 = m_2^2 = m_2^3 = m_2^4$,
$[(m_1^4 - m_1^3) > (=) (m_1^2 - m_1^1)] \rightarrow [f(\mathbf{m}^4) - f(\mathbf{m}^3) > (=) f(\mathbf{m}^2) - f(\mathbf{m}^1)]$.

The following proposition is now true.

Theorem 2. Let $f : M \rightarrow [0,1]$. Then, $\forall \mathbf{m} \in M$, $f(\mathbf{m}) = (m_1 + m_2)/2$ if and only if f satisfies Axioms I, II' and III.

The proof of Theorem 2 is here omitted: the result can be proved very much along the lines of the proof which Basu (1985) uses in order to justify Axiom N. (The proof is available with the author on request.)

4. *Variants of the Sen and the Foster-Greer-Thorbecke Poverty Measures*

If we preserve the axiomatic structure employed by Sen (1976) in the derivation of his poverty index, but replace his normalization axiom N by the normalization axiom N* advanced in this note, and write the general form of the poverty function as

$$P = A(n,q,z) + B(n,q,z)\sum_{i=1}^{q}(z - x_i)(q + 1 - i),$$

(where A and B are normalizing parameters and z, x_i, q and n are as already defined in Section 2 of this paper), then it can be shown that the poverty index implied by this axiomatic structure is given (for 'large' values of q) by

(4.1) $P^{S*} = [H + I + (1 - I)G^P]/2$.

A precise statement of the proposition, and its proof, are here omitted, since the required result can be routinely derived by mechanical application of the stratagem of proof employed by Sen (1976) to derive his poverty index P^S. (The theorem and its proof are available, on request, with the author.)

Notice now, by comparing (4.1) with (2.1), that when all poor incomes coincide with the poverty line, P^{S*} depends only on H – which is as it should be: unlike in the case of the Sen index, P^{S*} does not vanish whenever I becomes zero owing to all the poor having the poverty line income. While P^{S*} is obviously a very simple refinement of P^S, the refinement could conceivably be of some practical significance: specifically, it is important to note that P^{S*} can *reverse* the poverty ranking of distributions by P^S. An actual empirical example should help to highlight this. Subramanian (1988) furnishes some estimates of rural poverty in the Indian State of Tamil Nadu, and for the years 1961–62 and 1970–71 (here represented by the subscripts 1 and 2 respectively), he furnishes the following poverty-related information in terms of the triples (H_1 = .4434, I_1 = .2991, G_1^P = .1539) and (H_2 = .4967, I_2 = .2682, G_2^P = .1271)[3]. Assuming 'large' numbers of the poor, it is easy to verify that $P_1^S = .1804 > P_2^S = .1794$, but $P_1^{S*} = .4252 < P_2^{S*} = .4290$.

A similar refinement can be effected in the context of the Foster-Greer-Thorbecke (FGT)-related family of poverty indices P^*_α, given by:

(4.2) $$P_\alpha^* = (1/2n)\sum_{i=1}^{q}[1+((z-x_i)/z)^\alpha], \alpha \geq 0.$$

Three distinguished values of α are: $\alpha = 0$, $\alpha = 1$ and $\alpha = 2$ (see Foster, Greer and Thorbecke, 1984). By setting $\alpha = 0$, 1 and 2 respectively in (2.2), the FGT indices for these values of α can be seen to be given by: $P_0 = H$, $P_1 = HI$, and $P_2 = H[I^2 + (1 - I)^2 C_P^2]$, where C_P^2 is the square of the coefficient of variation in the distribution of income among the poor. For the same values of α, the values of the indices P^*_α (see (4.2)) are given by: $P_0^* = H$, $P_1^* = H(1 + I)/2$, and $P_2^* = H[1 + I^2 + (1 - I)^2 C_P^2]/2$. In general, for α greater than zero, P^*_α , unlike P_α, does not vanish when I is zero by virtue of all poor persons having the income z: it is an increasing function of the headcount ratio H for *every* value of the income-gap ratio I in the interval [0,1] – which is an attractive property not found in the P_α family of indices (α positive) when I is zero. Finally, for $\alpha > 0$, P^*_α and P_α can rank distributions in opposing ways. Again, an empirical illustration is afforded by poverty statistics for Finland. From estimates provided by Riihela, Sullstrom and Tuomala (2008), we can directly obtain or deduce[4], for two specific years, 2001 and 2004, the values of H, I and C_P^2, which are, respectively, .0160, .2625 and .0917 in 2001, and .0165, .2364 and .0705 in 2004. It is easily verified that P_2 has declined from .0019 in 2001 to .0016 in 2004, but P_2^* has increased from .00895 in 2001 to .00905 in 2004.

The story, however, does not quite end on a happy note. As pointed out by an anonymous referee, the measure P^{S*}, like the measure P^S, can violate the strong transfer axiom (which requires the poverty index to register an increase in value whenever a regressive transfer of income from any poor person to any other person occurs), while the measure P^*_α – *unlike* the measure P_α – can violate both strong transfer and restricted continuity (that is, continuity in poor incomes). The violation of strong transfer by P^*_α can be illustrated by means of an example supplied by the referee. Imagine two ordered 5-person income distributions given by **x** = (5,5,10,15,27) and **y** = (1,5,10,15,31), and suppose the poverty line to be given by $z = 30$. It is clear that **y** has been derived from **x** by a regressive (or upward) transfer of 4 units of income from the poorest individual to the richest one – a transfer which has also pushed the richest individual from below the poverty line to above it. It is easy to verify, for example, that $P_2(\mathbf{x}) = .4187 < P_2(\mathbf{y}) = .4647$, but $P_2^*(\mathbf{x}) = .7093 > P_2^*(\mathbf{y}) = .6323$: the transfer has *reduced* the valued of the poverty index P_2^*. That P^*_α can also fail continuity at the poverty line is reflected in the following: recalling that $P^*_\alpha = (1/2n)\sum_{i=1}^{q}[1+((z-x_i)/z)^\alpha], \alpha \geq 0$, note that if $x_i = z \forall i \in N$, then $P^*_\alpha = 1/2$, $\alpha \geq 0$; if now each person's income were to rise by an infinitesimal

[3] *The data can be found in Table 2 of Subramanian (1988). The poverty line is taken to be a consumption expenditure level of Rupees 15 per person per month at 1960-61 prices.*

[4] *The authors provide data on H, HI and P_2, from which the values of H, I and C_P^2 can be easily inferred. The estimates reproduced here relate to consumption poverty, and the poverty line employed is a relative one, pitched at 40 per cent of the median consumption level.*

amount ε, then P^*_α would plummet discontinuously to zero. Briefly, if strong transfer and continuity are regarded as indispensable properties for a poverty index, then, in terms of these properties, it would appear that P^{S*} is no better than P^S, while P^*_α is actually worse than P_α.

But are properties like strong transfer and continuity indeed unambiguously indispensable? This is questionable, as the following three lengthy quotations indicate. The first two – relating to the desirability of the continuity axiom – are due to Atkinson (1987; p.754) and Donaldson and Weymark (1986; p.674); and the third – relating to the desirability of the strong transfer axiom – is again due to Donaldson and Weymark (1986; p. 674):

> Here, there is room for difference of opinion. On the one hand, there are those who agree with Watts that there is a continuous gradation as one crosses the poverty line. On the other hand, there are people who see poverty as an either/or condition. A minimum income may be seen as a basic right, in which case the headcount [the archetypically discontinuous poverty function] may be quite acceptable as a measure of the number deprived of that right.

> Given the difficulties involved in measuring incomes accurately, it seems reasonable to require a poverty index to vary continuously with income. On the other hand, the use of a poverty line to sharply demarcate the rich from the poor suggests, but does not require, that a poverty index might be discontinuous at the poverty line. Thus, the *a priori* case for continuity is somewhat uneasy.

> Transfer principles are normally justified by the ethical appeal that similar principles have had in the measurement of inequality.With the strong transfer principles the arguments are less clearcut, since the number of poor people can change as a result of the transfer. An upward transfer may lower the number of poor people by pushing the richer poor person over the poverty line. At the same time, it increases the shortfall from the poverty line of the more destitute. These effects work in opposite directions, the first reducing poverty and the second increasing it (given monotonicity). It is true that inequality is increased by such a transfer, but it is not at all obvious that poverty is, at least if the poverty line represents a demarcation representing "absolute deprivation in terms of a person's capabilities" (Sen [1983]). Thus it may be desirable to permit poverty to decrease because of such a transfer in some cases, and [strong upward transfer] must be rejected. Sen [1976] himself suggested that [strong upward transfer] was a reasonable property of a poverty index, but subsequently advocated [weak transfer] instead. The strong transfer principles have been a source of controversy ever since.

5. *Concluding Observations*

This paper amounts to one more confirmation of the proposition that the measurement of poverty can be a complicated enterprise. It suggests that one may not always be free to indifferently adopt a strong or weak definition of the poor. Whether or not this suggestion is a convincing one, the paper proceeds, in a positive spirit, to consider one specific and troublesome implication of adopting a strong, or inclusive, definition. This implication, for a number of extant measures of poverty, is that in the special case in which all the poor individuals in a society have the poverty line income, measured poverty fails to be a monotonically increasing function of the headcount ratio. The paper advances alternative versions of the Sen and Foster-Greer-Thorbecke poverty measures as a means of avoiding this problem. In the case of the Sen index, the problem is traced to Sen's normalization axiom, and an alternative normalization axiom is advanced and rationalized. In the process of conducting this exercise, the paper also reviews and draws on Basu's decomposition of Sen's normalization axiom. The variants of the Sen and Foster-Greer-Thorbecke indices may circumvent the specific problem which motivated their quest, but the latter set of variants, unlike the corresponding originals, fall foul of certain axioms that have been proposed in the literature, such as the strong transfer property and the continuity property. These properties are not, it can be argued, self-evidently and completely compelling. Clearly, something must give, and it must be left to the analyst to effect the tradeoffs on the basis of her values, priorities, and perceptions. It is hoped that the problem of choice which is entailed is at least not a wholly trivial one.

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