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AN N-PERSON BARGAINING GAME*

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Rubinstein type bargaining is extended to n-person case with linear utilities. The game possesses a unique subgame perfect equilibrium in which everybody's share is equal as impatience goes to zero. (JEL C78)

1. Introduction

We study an n-player bargaining game which can be regarded as an extension of Rubinstein's (1982) two player bargaining game. The players bargain over a cake of unit size. In this game players' order is important. Being player 1 signifies that nobody can get his share before him. Similarly, only players with indices less than i can get their share before player i. This can be interpreted as bargaining over the surplus of a joint project in which players $i+1$, $i+2,\ldots, n$ cannot do anything without player i, $i = 1, 2,\ldots, n-1$. In the game constructed here there is a unique subgame perfect equilibrium (SPE). The uniqueness result stems from the fact that the game is not a unanimity game. In n-player bargaining games where the players have to be unanimous about the division there typically exist a plethora of SPE. The equilibria are supported by the threat that a player who rejects a deviant offer is rewarded by giving him the whole cake next period.

The unique SPE of the game we construct here exhibits several attractive features. The agreement over the division of the cake is reached in the first period. When all the players have the same discount factor δ the first player's share is $x^* = 1/\left[1 + \delta + \delta^2 + \ldots + \delta^{n-1}\right]$, the second player's share is δx^* and the share of the n^{th} player is δ times that of the $n-1^{th}$ player. This extends the first mover advantage of Rubinstein's model to a previous mover advantage. As δ approaches unity every player's share approaches one n^{th} of the total cake. With two players the game reduces to the original game. Jun (1987), Chae and Yang (1988, 1990 and 1992) and Serrano (1990) have presented related models.

2. The Model

There are n players who try to divide a cake of size unity. In the first period player 1 tells player 2 what share of the cake he wants. If player 2 accepts, player 1 gets the share he demanded and then it is player 2's turn to make a similar demand to player 3. As long as players accept the demand of the previous player everything takes place within the same period. In case player 2 rejects player 1's demand the game proceeds to the next period. In this period player 2 offers a share to player 1. If player 1 accepts the offer rest of the cake will be divided between players 2, 3, ..., n in a game which starts in the same period and in which player 2 first demands a share. If player 1 rejects player 2's offer the game proceeds to the next period and player 3 offers a share to player 1. If the

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offer is accepted the remaining cake will be divided between players 2, 3, ..., n in a game that starts with player 2 making a demand to player 3. Rejection always causes transition to the next period in which the next player makes an offer.

At any stage the player with the lowest index is the first one who can get away. The players offer a share of the cake to the player with the lowest index in turns. When it is his turn he makes a demand, of course, and it is the player with the second lowest index who either accepts or rejects the demand. When one player gets away with his share the subsequent game starts by a demand of the player with the lowest index. Formally, denote by G_n (z;i) an n-player game with a cake of size z, and with player i making an offer. Assume he offers a share x of the cake. If the offer is accepted the continuation game immediately becomes G_{n-1} (z–x; 2). In this game the size of the cake is $z–x$, player 2 has the lowest index and he makes a demand that player 3 can accept or reject. If the offer x by player i in game G_n (z; i) is rejected the continuation game becomes G_n (z; i + 1) and it starts in the next period. In this game player i + 1 makes an offer to player 1. More generally if there are $n-k$ players left, that is players k + 1, k + 2,..., n, and player $k + m$ makes an offer to player $k + 1$ the game is denoted by G_{n-k} (z; k + m). Assume he offers a share x of the cake. If the offer is accepted the continuation game in the same period is denoted by G_{n-k-1} (z–x; k + 2). If the offer is rejected the continuation game starting next period is denoted by G_{n-k} (z; k + m + 1).

Theorem. There exists a unique subgame perfect equilibrium (SPE) in the above n-player bargaining game.

We only sketch the proof since it is analogous to that presented in Osborne and Rubinstein (1990).

First a subgame perfect equilibrium is constructed and then it is shown to be unique. Consider a game with n players and a strategy according to which player 1 always demands w_1 (n) and accepts an offer $x \geq w_i$ (n) of player i, player 2 always offers w_2 (n) and accepts a demand $x \leq w_1$ (n) of player 1, and player i always offers w_i (n), i = 3,..., n. Note that for the moment we consider only that part of the strategies that applies to the game with n players present. When there are less than n players left the remaining players use analogous strategies. The equations that make player 2 indifferent between accepting player 1's demand and offering him w_2 (n), and player 1 indifferent between receiving w_i (n) this period and w_{i+1} (n), i = 2, 3,...,n, one period later are as follows.

$$(1) \qquad 1-w_1(n) = \delta\left[1-w_2(n)\right]$$

$$(2) \qquad w_i(n) = \delta w_{i+1}(n),\ i = 2,3,\ldots,n.$$

When there are n players the additions are modulo n, i.e. $n + 1 = 1$.

It is easy to see that $w_1(n) = [1 + \delta + \ldots + \delta^{n-1}]^{-1}$. The same procedure can then be applied to the continuation game with $n-1$ players and so on. As the above strategies are determined by equations (1) and (2) it can be shown in a standard manner that they constitute a SPE in which the shares of players 1, 2,...,n are respectively $x^*, \delta x^*, \delta^2 x^*, \ldots, \delta^{n-1}x^*$, where $x^* = \left[1 + \delta + \ldots + \delta^{n-1}\right]^{-1}$.

The uniqueness of the above SPE is shown in the usual way. Denote by $G_n(i)$ a subgame that starts with an offer of player i when there are n players left in the game. The existence of a SPE allows the following definitions.

$M_n(1)$ = sup {present value of x_1 for player 1: there is a SPE of $G_n(1)$ where x_1 is player 1's share}

$m_n(1)$ = inf {present value of x_1 for player 1: there is a SPE of $G_n(1)$ where x_1 is player 1's share}

$M_n(i)$ = sup {present value of $1 - x_i$ for player i: there is a SPE of $G_n(i)$ where $1 - x_i$ remains to be divided between players 2, 3, ..., n}

$m_n(i)$ = inf {present value of $1 - x_i$ for player i: there is a SPE of $G_n(i)$ where $1 - x_i$ remains to be divided between players 2, 3, ..., n}

i = 2, 3, ...,n

The present value is calculated regarding the beginning of subgame $G_n(i)$, i = 1, 2, ..., n, as the first period for player i.

It will be shown that $M_n(i) = m_n(i)$ for all i = 1, 2, ..., n. The following $2n$ inequalities are sufficient to establish the fact.

$$(3) \quad M_n(1) \leq 1 - \delta m_n(2)$$

$$(4) \quad m_n(1) \geq 1 - \delta M_n(2)$$

(5) $M_n(i) \leq 1 - \delta + \delta M_n(i+1)$

(6) $m_n(i) \geq \min\{1 - \delta + \delta m_n(i+1), \ldots, 1 - \delta^{n-i} + \delta^{n-i} m_n(n), 1 - \delta^{n-i+1} M_n(1)\}$
$i = 2,3,\ldots,n-1$

(7) $M_n(n) \leq 1 - \delta m_n(1)$

(8) $m_n(n) \geq 1 - \delta M_n(1)$.

For players 1 and n the inequalities are similar to those of the two-player case (Osborne and Rubinstein, 1990, p. 47). In (5) player 1 rejects the offer x of player i if $x < \delta\left[1 - M_n(i+1)\right]$. In (6) player 1 would accept the offer x if x is greater than the greatest of $\delta[1 - m_n(i+1)], \ldots, \delta^{n-i}\left[1 - m_n(n)\right]$, and $\delta^{n-i+1} M_n(1)$. Starting from value $n-1$ one can substitute in (6) the infimum from (8). Then using (5), (6), (7) and (8) and repeated substitution the inequalities of players i, $i = 2, 3, \ldots, n-1$ reduce to $M_n(i) \leq 1 - \delta + \delta M_n(i+1)$ and $m_n(i) \geq 1 - \delta^{n-i+1} M_n(1)$. Now it is easy to establish that $m_i(n) = M_i(n)$.

The case above refers to a situation in which all the n players are still left in the game. But the same arguments and formulae (slightly modified) can be used for the case in which there are $n-k$ players in the game. This shows the uniqueness of the SPE. Implicit in this sketch of a proof is its inductive nature. Since the solution of the two player game is known, it is clear that the proof could be made by induction on the number of players remaining in the game. To do this formally requires quite heavy notation and this would not add very much to the understanding of the results.

3. Summary

In this paper we constructed an n-player bargaining game that has several features that are analogous to the two player bargaining game of Rubinstein (1982), There is a unique SPE in which agreement is achieved in the first period. The players have a previous mover advantage. As the players become more patient the outcome of the game approaches equal division. With only two players it is not possible to differentiate between unanimity and nonunamity games. With more than two players this distinction becomes crucial. The game in this paper is a nonunanimity game and it is this feature that is behind the uniqueness of the SPE.

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