Finnish Economic Papers – Volume 6 – Number 2 – Autumn 1993

ON THE USE OF THE BLACK & SCHOLES MODEL IN A STOCHASTIC INTEREST RATE ECONOMY

KRISTER RINDELL

Swedish School of Economics, Department of Finance,
Arkadiagatan 22,
FIN-00100 Helsinki, Finland

This paper examines the performance of the Black & Scholes (1973) model for pricing of European style stock options in a stochastic interest rate economy. Throughout the paper we assume that Jarrow's (1988) version of the Merton (1973) model correctly describes the reality. We examine the implications of two standard estimation methods of the value of the volatility parameter in the Black & Scholes model, the historical estimate method and the implied value method, respectively. Specific formulae are given in order to determine whether the Black & Scholes model under- or overprices options. Numerical examples show that, in some cases, the pricing error can be sizeable even for short term options. (JEL G13).

1. Introduction

Embedding stochastic interest rates in pricing models of asset price risk contingent claims is clearly in vogue among financial economists. The work was pioneered by Merton (1973), who extended the Black & Scholes (1973) model to stochastic interest rates. Recent contributions include Cheng (1991), Jarrow (1988), Kishimoto (1989) and Turnbull & Milne (1991). Of these, the Kishimoto and Jarrow models can easily be analysed in the sense that they provide specific parameterizations of the asset and bond price dynamics. The models are closely related, the Jarrow model being the continuous time version of the Kishimoto model.

The purpose of this paper is to analyse the lognormal version of the Jarrow model[1] for pricing of European call options by relating it to its constant interest rate counterpart, the popular Black & Scholes model. We try to determine whether the effect of stochastic interest rates is large enough to motivate the use of the Jarrow model instead of the much simpler Black & Scholes model. Two questions will be addressed. First, what is the pricing error if the Black & Scholes model is used in the stochastic interest rate economy described in section two. In order to use the Black & Scholes model one needs to estimate the stock price volatility. Two commonly used methods to obtain the volatility will be considered, namely measuring the volatility from historical price data and the use of implied volatilities. Second, how sensitive are prices of call options to changes in the interest rate volatility. This is analyzed by deriving comparative statics of the Jarrow model. All the results in sections 3–4 are original.[2]

[1] *To be exact, we examine a constant volatility, two factor version of the model.*

[2] *With the exception of the comparative statics of the Black & Scholes model, which are included for completeness.*

2. The models

To derive the Jarrow model assume that

[A1] Markets are frictionless and free from arbitrage opportunities,
[A2] Trading takes place continuously,
A[3] A continuum of riskless discount bonds with maturities $t \in [0, T]$ are available,
[A4] Stock prices follow Ito processes,

$$(1)\quad \frac{dS(t)}{S(t)} = \alpha dt + \sigma_1 dw_1(t) + \sigma_2 dw_2(t),$$

where
$S(t)$ is the price of the stock at time t,
α is the stock's expected instantaneous rate of return,
σ_1 is the instantaneous asset price risk sensitivity, a positive constant,
σ_2 is the instantaneous interest rate risk sensitivity, a constant,
$dw(t)$ is an increment to a Brownian motion,

[A5] Instantaneous forward rates[3] follow Ito processes in which for a fixed but arbitrary τ, $\tau \in [0, T]$,

$$(2)\quad df(t,\tau) = \mu dt + \delta dw_2(t),$$

where
$f(t,\tau)$ is the instantaneous forward rate for time τ at time t, $t \in [0, T]$,
μ the instantaneous expected change in $f(t,\tau)$ at time t,
δ the instantaneous standard deviation of the change in $f(t,\tau)$ at time t, a positive constant.

The assumptions state that stock returns are generated by a two factor model. The first factor induces asset price risk, i.e. risk specific for non-default free securities such as common stock. The second factor is the interest rate factor which, by definition, is orthogonal to the first factor. The asset price risk and the interest rate risk sensitivities of the stock are given by σ_1 and σ_2, respectively. Changes in the forward rates are generated by a one factor model, the factor being of course the interest rate factor. The interest rate risk sensitivity of the forward rates is given by δ, a positive constant. Note that σ_2, the interest rate risk sensitivity of the stock, takes either a positive or a negative value depending on whether the covariance between stock returns and changes in interest rates is positive or negative.

Given the assumptions above the price of a European call option can be calculated. The Black & Scholes model is obtained by setting the interest rate factor to zero and assuming that there is no drift in forward rates. Below we give the Jarrow and the Black & Scholes formulas, respectively, and, because both are special cases of it, the Merton formula for pricing of European call options. The price of a European call option at time zero, with an exercise price K and time-to-maturity T is

$$(3)\quad C(0,K,T) = S(0)N(h) - KP(0,T)N(h-\xi),$$

$$h = \frac{\ln\left[\frac{S(0)}{KP(0,T)}\right] + \frac{1}{2}\xi^2}{\xi},$$

where, depending on the model,

MERTON

$$\xi_M^2 = \int_0^T \left\{ VAR\left[\ln\left(\frac{S_t}{S_{t-dt}}\right)\right] - 2COV \left[\ln\left(\frac{S_t}{S_{t-dt}}\right), \ln\left(\frac{P(t,T)}{P(t-dt,T)}\right)\right](t) + VAR\left[\ln\left(\frac{P(t,T)}{P(t-dt,T)}\right)\right](t) \right\} dt,$$

BLACK & SCHOLES

$$P(0,T) = \exp\{-rT\},$$
$$\xi_{BS}^2 = \sigma^2 T,$$

JARROW

$$P(0,T) = \exp\left\{-\int_0^T f(0,y)\,dy\right\},$$

$$\xi_J^2 = \left[(\sigma_1^2 + \sigma_2^2)T + \delta\sigma_2 T^2 + \frac{1}{3}\delta^2 T^3\right].$$

[3] *An instantaneous forward rate is defined as* $f(t,\tau) = -\frac{\partial P(t,\tau)/\partial\tau}{P(t,\tau)}$, *where* $P(t,\tau)$ *is the current price of a zero coupon bond with maturity at time* τ. *The instantaneous spot rate,* $r(t)$, *is given by* $f(t,t)$.

where
S(0) is the current stock price,
r is the constant annual interest rate,
P(0,T) is the current price of a zero coupon bond with maturity at time T,
σ is the instantaneous standard deviation of the stock's rate of return, a constant,
N() is the cumulative standard normal distribution function.

The instantaneous covariance between stock returns and forward rate changes can be shown to be[4]

$$(4) \quad \mathrm{Cov}[dS/S, df(t,\tau)] = \sigma_2 \delta dt,$$

and the corresponding correlation coefficient

$$(5) \quad \rho = \frac{\sigma_2^2}{\sqrt{\sigma_1^2 + \sigma_2^2}}.$$

For example, a negative correlation between stock returns and changes in forward rates obtains when σ_2 is negative. This implies a positive correlation between stock and bond prices due to bond prices being negatively correlated with interest rates.

In order to use the Jarrow model one has to estimate three different parameter values, $\{\sigma_1, \sigma_2, \delta\}$, compared to only one, σ, in the Black & Scholes model. Therefore, if it is used, the improvement in the accuracy of the theoretical prices should outweigh the trouble of estimating the two extra parameter values. The improvement is of course related to how big the difference between the option prices given by the respective models is. This question will be addressed in the next section.

3. *The effect of stochastic interest rates on call option prices*

In this section the extent to which the Black & Scholes formula misprices options in a stochastic interest rate economy will be analyzed. Throughout the section we assume that the asset price dynamics are given by expressions (1) and (2), that is that the Jarrow model is the correct model for pricing of options.

[4] *Use the definition of covariance and multiplication rules of ITO calculus, see e.g. Karatzas I. & Schreve S.E. (1988).*

The user of the Black & Scholes model has to determine two parameter values, the constant interest rate, r, and the stock price volatility, σ. The constant interest rate is usually approximated with the yield of a matched maturity zero coupon bond. As seen from expression (3), the only difference between the Black & Scholes and the Jarrow models shows up in the total volatilities, ξ^2. Consequently, if the total volatilities do not differ, neither do the option prices, that is

$$\xi_{BS}^2 = \xi_J^2 \Rightarrow C_{BS} = C_J,$$

where C_{BS} denotes the call price given by the Black & Scholes formula and C_J the call price given by the Jarrow formula. The user has much more variety in estimating a value for the stock price volatility, σ. There are, in principle, two ways to estimate the volatility. It can be estimated from historical price data, denoted by σ_{HIST}, or implied from market prices of options, denoted by σ_{IMP}. Next, the pricing errors resulting from these estimation methods are analyzed.

3.1 *Historical volatility estimates and the pricing error*

The commonly used historical estimate of stock price volatility is the annual standard deviation of logarithmic price relatives, that is $\sqrt{\sigma_1^2 + \sigma_2^2}$. Then

$$(6) \quad \xi_{BS}^2(\mathrm{HIST}) = \sigma_{HIST}^2 T = (\sigma_1^2 + \sigma_2^2)T, \forall T.$$

The method captures the stock return variance part in ξ_J^2, in equation (3). However, the direct effect of the interest rate volatility, that is the variance of bond returns, and the covariance effect are ignored. Consequently

$$(7) \quad \xi_{BS}^2(\mathrm{HIST}) \neq \xi_J^2.$$

This leads to that the call option price given by the Black & Scholes formula will in general differ from the one given by the Jarrow formula. Two questions arise. First, what is the direction of the error, that is does the Black & Scholes formula over- or underprice options.

Second, how significant, in money terms, is the error.

Whether option prices increase or decrease when stochastic interest rates are introduced depends on the level and the sign of the covariance between stock and bond returns. Clearly, as seen from expression (3), if the covariance is negative option prices always increase. On the other hand, if the covariance is positive the effect is ambiguous. However, it can easily be shown, by using expression (3), that a sufficient condition for under/overpricing of the Black & Scholes model is

$$(8) \quad -\sigma_2 \lesseqgtr \frac{1}{3}\delta T \Rightarrow \xi_J^2 \gtreqless \xi_{BS}^2(\text{HIST}) \Rightarrow C_J \gtreqless C_{BS}.$$

Alternatively, by using expression (5), a »critical» level of correlation between stock returns and interest rate changes, ρ_c, can be calculated. It is defined as the level of correlation that separates under/overpricing of options and is given by

$$(9) \quad \rho_c = -\frac{1}{3}\frac{\delta T}{\sqrt{\sigma_1^2+\sigma_2^2}}.$$

And hence,

$$(10) \quad \rho \gtreqless \rho_c \Rightarrow \xi_J^2 \gtreqless \xi_{BS}^2(\text{HIST}) \Rightarrow C_J \gtreqless C_{BS}.$$

Note that, given the parameter values, (8) and (10) hold with equality only for one time-to-maturity. The expressions can relatively easily be studied with time series data on stock prices and interest rates and it should be the first step in implementing the models. If one is fortunate the pricing error is small and the simpler Black & Scholes model can be used.

Example 1:

A numerical example is always illustrative. We consider typical short-term call options traded on financial markets and a relatively wide range of covariances between asset returns and interest rates. The parameter values used to calculate the option prices are:

$$S = 500 \quad K \in \{400, 450, 500, 550, 600\}$$

$$T \in \{.25, .75\} \qquad \frac{-\ln P(.)}{T} = .1$$

$$\sigma_1 = .20 \qquad \sigma_2 \in \{-.1, -.05, 0, .05, .1\} \qquad \delta = .05.$$

The volatility coefficients σ_1 and δ are adapted from Rindell (1991). The corresponding Black & Scholes volatilities, $\sqrt{\sigma_1^2+\sigma_2^2}$, vary between .2 and .22. The correlation coefficients, given by equation (5), vary between –.45 and .45. Finally the critical correlation coefficients vary between –.06 and –.02. The resulting option values and their corresponding delta gamma values are given in table 1.[5]

The table shows that the price differences can be rather sizable, even for short-term options. The differences are large especially for out of the money options with high correlation coefficients, close to 7 % for the 3 month options and close to 9 % for the 9 month options. Even for at the money options the price difference can be over 2 percentage points. For strongly in the money options the differences are quite small. Interestingly the delta and the gamma values are virtually identical. This means that even though a trader might use incorrect *prices* if he used the Black & Scholes model he would still choose the correct *number* of options to change his portfolio risk.

3.2 Implied volatility estimates and the pricing error

When the stock price volatility is implied, the Black & Scholes model value is equated with an observed market price of an option by choosing an appropriate value for σ, denoted by σ_{IMP}. That is, for some fixed time-to-maturity of the option, say T^*, the volatility is chosen such that

$$(11) \quad \xi_{BS}^2(\text{IMP}) = \sigma_{IMP}^2 T^* = [(\sigma_1^2+\sigma_2^2)T^* + \delta\sigma_2 T^{*2} + \frac{1}{3}\delta^2 T^{*3}] = \xi_J^2.$$

[5] *Delta is the partial derivative of the option price with respect to the underlying asset price. Gamma is the corresponding second partial derivative. These measures are used to determine the number of options one wants to buy/sell in order to change the risk profile of one's portfolio.*

OPTION VALUES

Jarrow model, T=.25, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.70	0.45	0.39	0.49	0.80
450	7.02	5.89	5.57	6.09	7.42
500	28.39	26.84	26.40	27.13	28.93
550	66.22	65.34	65.10	65.50	66.55
600	112.90	112.64	112.58	112.68	113.01

Jarrow model, T=.75, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	9.58	8.17	8.01	9.12	11.42
450	27.09	25.01	24.77	26.41	29.68
500	56.07	54.01	53.78	55.40	58.65
550	94.44	92.92	92.75	93.94	96.45
600	138.83	137.92	137.82	138.52	140.11

Black & Scholes model, T=.25, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.75	0.47	0.39	0.47	0.75
450	7.21	5.98	5.56	5.98	7.21
500	28.65	26.98	26.39	26.98	28.65
550	66.38	65.41	65.10	65.41	66.38
600	112.95	112.66	112.57	112.66	112.95

Black & Scholes model, T=.75, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	10.39	8.53	7.90	8.53	10.39
450	28.25	25.54	24.59	25.54	28.25
500	57.22	54.54	53.60	54.54	57.22
550	95.33	93.30	92.62	93.30	95.33
600	139.38	138.14	137.75	138.14	139.38

DELTA

Jarrow model, T=.25, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.04	0.03	0.03	0.03	0.04
450	0.25	0.23	0.22	0.23	0.26
500	0.61	0.61	0.62	0.61	0.61
550	0.87	0.89	0.89	0.89	0.87
600	0.97	0.98	0.98	0.98	0.97

Jarrow model, T=.75, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.24	0.22	0.22	0.23	0.26
450	0.47	0.46	0.46	0.46	0.47
500	0.68	0.69	0.69	0.69	0.68
550	0.84	0.85	0.85	0.84	0.82
600	0.93	0.94	0.94	0.93	0.91

Black & Scholes model, T=.25, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.04	0.03	0.03	0.03	0.04
450	0.25	0.23	0.22	0.23	0.25
500	0.61	0.61	0.62	0.61	0.61
550	0.87	0.89	0.89	0.89	0.87
600	0.97	0.98	0.98	0.98	0.97

Black & Scholes model, T=.75, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.25	0.22	0.22	0.22	0.25
450	0.47	0.46	0.46	0.46	0.47
500	0.68	0.69	0.69	0.69	0.68
550	0.83	0.85	0.85	0.85	0.83
600	0.92	0.94	0.94	0.94	0.92

GAMMA

Jarrow model, T=.25, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.09	0.07	0.06	0.07	0.09
450	0.32	0.30	0.30	0.31	0.32
500	0.38	0.38	0.38	0.38	0.38
550	0.21	0.19	0.18	0.19	0.21
600	0.06	0.05	0.04	0.05	0.07

Jarrow model, T=.75, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.31	0.30	0.29	0.31	0.32
450	0.40	0.40	0.40	0.40	0.40
500	0.36	0.35	0.35	0.35	0.36
550	0.24	0.23	0.23	0.24	0.26
600	0.14	0.12	0.12	0.13	0.16

Black & Scholes model, T=.25, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.09	0.07	0.06	0.07	0.09
450	0.32	0.31	0.30	0.31	0.32
500	0.38	0.38	0.38	0.38	0.38
550	0.21	0.19	0.18	0.19	0.21
600	0.06	0.05	0.04	0.05	0.06

Black & Scholes model, T=.75, σ2

K	-0.10	-0.05	0.00	0.05	0.10
400	0.32	0.30	0.29	0.30	0.32
450	0.40	0.40	0.40	0.40	0.40
500	0.36	0.35	0.35	0.35	0.36
550	0.25	0.24	0.23	0.24	0.25
600	0.15	0.13	0.12	0.13	0.15

Table 1. Option values and their corresponding delta and gamma values.

For this specific time-to-maturity the pricing error, is by construction, zero. However, for all other times-to-maturity there is a pricing error, that is

$$(12) \quad \begin{cases} \xi^2_{BS}(\text{IMP}) = \xi^2_J \text{ if } T = T^* \\ \xi^2_{BS}(\text{IMP}) \neq \xi^2_J \text{ if } T \neq T^* \end{cases}.$$

The questions to be addressed are the direction and degree of the pricing error as T moves away from T*. For the analysis we need a preliminary result, namely the functional form of the total volatility in the Jarrow model, ξ^2_J, with respect to the time-to-maturity. It can be shown, by using equation (3), that ξ^2_J is concave/convex in time-to-maturity if $-\sigma_2 \gtrless \delta T$.[6] Further, recall that the total volatility in the Black & Scholes formula is always linear, and monotonically increasing, in time-to-maturity. Consequently, the direction of the pricing error can be shown to be

$$(13) \quad \begin{cases} T \lesseqgtr T^* \Rightarrow \xi^2_J \gtreqless \xi^2_{BS}(\text{IMP}) \Rightarrow \\ C_J \gtreqless C_{BS} \text{ if } -\sigma_2 > \delta T \\ T \lesseqgtr T^* \Rightarrow \xi^2_J \lesseqgtr \xi^2_{BS}(\text{IMP}) \Rightarrow \\ C_J \lesseqgtr C_{BS} \text{ if } -\sigma_2 < \delta T \end{cases}.$$

Example 2:

Using the same values for K, S, σ_1, σ_2, δ and r as in Example 1 a set of option prices according to the Jarrow model was calculated for a time to maturity of 6 months. The implied volatility for the Black & Scholes model was calculated using these prices and the corresponding Black & Scholes prices were calculated. These were by construction equal to the Jarrow prices. To study resulting pricing er-

[6] *This is due to the fact that* $\frac{\partial \xi^2_J}{\partial T} > 0$ *and* $\frac{\partial^2 \xi^2_J}{\partial T^2} \gtrless 0$ *as* $\sigma_2 + \delta T \gtrless 0$.

OPTION VALUES

Jarrow model

T=.17

K	$\sigma 2$: -0.10	-0.05	0.00	0.05	0.10
400	0.00	0.00	0.00	0.00	0.00
450	0.07	0.04	0.03	0.04	0.07
500	9.70	9.03	8.80	9.05	9.74
550	52.01	51.96	51.95	51.97	52.01
600	101.92	101.92	101.92	101.92	101.92

T=.5

K	$\sigma 2$: -0.10	-0.05	0.00	0.05	0.10
400	4.36	3.44	3.26	3.80	5.11
450	17.30	15.50	15.13	16.23	18.66
500	43.43	41.51	41.11	42.29	44.89
550	81.27	79.93	79.66	80.46	82.33
600	126.24	125.57	125.45	125.83	126.82

T=1

K	$\sigma 2$: -0.10	-0.05	0.00	0.05	0.10
400	15.42	13.74	13.74	15.42	18.62
450	36.32	34.17	34.17	36.32	40.30
500	67.36	65.31	65.31	67.36	71.20
550	106.36	104.79	104.79	106.36	109.43
600	150.56	149.54	149.54	150.56	152.71

OPTION VALUES

Black & Sholes model

T=.17

K	$\sigma 2$: -0.10	-0.05	0.00	0.05	0.10
400	0.00	0.00	0.00	0.00	0.00
450	0.06	0.03	0.03	0.04	0.09
500	9.52	8.94	8.82	9.18	9.95
550	51.99	51.96	51.95	51.97	52.03
600	101.92	101.92	101.92	101.92	101.92

T=.5

K	$\sigma 2$: -0.10	-0.05	0.00	0.05	0.10
400	4.36	3.44	3.26	3.80	5.11
450	17.30	15.50	15.13	16.23	18.66
500	43.43	41.51	41.11	42.29	44.89
550	81.27	79.93	79.66	80.46	82.33
600	126.24	125.57	125.45	125.83	126.82

T=1

K	$\sigma 2$: -0.10	-0.05	0.00	0.05	0.10
400	16.04	13.95	13.52	14.80	17.64
450	37.10	34.45	33.89	35.53	39.10
500	68.10	65.58	65.05	66.60	70.03
550	106.94	104.99	104.59	105.77	108.48
600	150.95	149.66	149.41	150.17	152.02

Table 2. Options values using implied volatilities for the B & S model.

rors when the time to maturity changes, new prices were calculated for times to maturity of 2 months and 12 months, respectively.

The results are shown in Table 2. Again the pricing errors are largest for the out of the money options with a high σ_2. For at the money options the pricing error does not exceed 2 %.

4. *The effect of changes in volatility and time-to-maturity on call option prices*

In this section the use of the Jarrow model is discussed. More specifically, if the model is used one needs estimates of three parameter values, $\{\sigma_1, \sigma_2, \delta\}$, which leads to a risk of substantial estimation erros.[7] To examine how sensitive option prices are to estimation errors in these parameter values we derive comparative statics with respect to them. After that, to complete the section, comparative statics with respect to the time-to-maturity of the option are derived.

The sensitivities of the Jarrow formula to changes in the volatility coefficients are:

$$\text{(14)} \quad \frac{\partial C_J}{\partial \sigma_1} \equiv \eta_1 = SN'(h)\frac{\sigma_1 T}{\xi_J} > 0,$$

$$\text{(15)} \quad \frac{\partial C_J}{\partial \sigma_2} \equiv \eta_2 = SN'(h)\frac{(\sigma_2 + \frac{1}{2}\delta T)\,T}{\xi_J}$$

$$\begin{cases} > 0 \text{ if } \sigma_2 + \frac{1}{2}\delta T > 0 \\ = 0 \text{ if } \sigma_2 + \frac{1}{2}\delta T = 0 \\ < 0 \text{ if } \sigma_2 + \frac{1}{2}\delta T < 0 \end{cases},$$

$$\text{(16)} \quad \frac{\partial C_J}{\partial \delta} \equiv \eta_3 = SN'(h)\frac{(\frac{1}{2}\sigma_2 + \frac{1}{3}\delta T)\,T^2}{\xi_J}$$

$$\begin{cases} > 0 \text{ if } \frac{1}{2}\sigma_2 + \frac{1}{3}\delta T > 0 \\ = 0 \text{ if } \frac{1}{2}\sigma_2 + \frac{1}{3}\delta T = 0 \\ < 0 \text{ if } \frac{1}{2}\sigma_2 + \frac{1}{3}\delta T < 0 \end{cases},$$

where a prime denotes a derivative and ∂ denotes a partial derivative. As the asset price

7 *Of course, potential estimation errors in the Black & Scholes model should not be ignored either.*

Jarrow model

$\eta 1$

T=.25

$\sigma 2$

		-0.10	-0.05	0.00	0.05	0.10
	400	15.86	13.13	12.31	13.66	16.73
	450	64.71	66.84	67.40	66.45	63.93
K	500	86.85	93.34	95.38	92.05	84.81
	550	51.76	50.71	50.24	50.97	51.96
	600	16.85	13.63	12.67	14.25	17.90

$\eta 1$

T=.75

$\sigma 2$

		-0.10	-0.05	0.00	0.05	0.10
	400	99.23	101.26	101.47	99.94	96.28
	450	143.37	152.62	153.76	146.26	133.27
K	500	142.45	150.43	151.39	144.97	133.51
	550	107.81	108.87	108.94	108.23	105.70
	600	66.65	62.65	62.12	65.46	70.27

Jarrow model

$\eta 2$

T=.25

$\sigma 2$

		-0.10	-0.05	0.00	0.05	0.10
	400	-7.43	-2.87	0.38	3.84	8.89
	450	-30.33	-14.62	2.11	18.69	33.96
K	500	-40.71	-20.42	2.98	25.89	45.05
	550	-24.26	-11.09	1.57	14.33	27.60
	600	-7.90	-2.98	0.40	4.01	9.51

$\eta 2$

T=.75

$\sigma 2$

		-0.10	-0.05	0.00	0.05	0.10
	400	-40.31	-15.82	9.51	34.35	57.17
	450	-58.24	-23.85	14.41	50.28	79.13
K	500	-57.87	-23.50	14.19	49.83	79.27
	550	-43.80	-17.01	10.21	37.20	62.76
	600	-27.08	-9.79	5.82	22.50	41.72

Jarrow model

$\eta 3$

T=.25

$\sigma 2$

		-0.10	-0.05	0.00	0.05	0.10
	400	-3.63	-1.37	0.26	1.99	4.53
	450	-14.83	-6.96	1.40	9.69	17.31
K	500	-19.90	-9.72	1.99	13.42	22.97
	550	-11.86	-5.28	1.05	7.43	14.07
	600	-3.86	-1.42	0.26	2.08	4.85

$\eta 3$

T=.75

$\sigma 2$

		-0.10	-0.05	0.00	0.05	0.10
	400	-18.61	-6.33	6.34	18.74	30.09
	450	-26.88	-9.54	9.61	27.42	41.65
K	500	-26.71	-9.40	9.46	27.18	41.72
	550	-20.21	-6.80	6.81	20.29	33.03
	600	-12.50	-3.92	3.88	12.27	21.96

Black & Scholes model

η

T=.25

$\sigma 2$

		-0.10	-0.05	0.00	0.05	0.10
	400	18.21	13.79	12.28	13.79	18.21
	450	71.93	68.71	67.42	68.71	71.93
K	500	95.99	95.60	95.43	95.60	95.99
	550	57.99	52.40	50.23	52.40	57.99
	600	19.41	14.34	12.65	14.34	19.41

η

T=.75

$\sigma 2$

		-0.10	-0.05	0.00	0.05	0.10
	400	109.52	103.89	101.62	103.89	109.52
	450	155.05	154.76	154.63	154.76	155.05
K	500	154.66	152.88	152.13	152.88	154.66
	550	119.58	112.01	108.99	112.01	119.58
	600	76.52	65.74	61.71	65.74	76.52

Table 3. Partial derivatives of the option values wrt the volatility coefficients.

risk related volatility increases the value of the call option increases. The effect of increases in the interest rate related volatilities σ_2 and δ, is ambiguous if the covariance between the stock and bond returns is positive, that is if σ_2 is negative. This is because as either one is increased the variance and the covariance effects work in opposite directions in ξ_J^2. The sign of the derivative depends on which of these effects dominates. There are even cases in which these two effects cancel out, and there is no change in the call value.

As a comparison, the sensitivity of the Black & Scholes formula to a change in the instantaneous volatility is

$$(17) \quad \frac{\partial C_{BS}}{\partial \sigma} \equiv \eta = SN'(h)\sqrt{T} > 0.$$

Example 3:

Table 3 gives the partial derivatives based on parameter values in Example 1. The ones related to interest rate risk are at their highest when the option is at the money and the correlation coefficient between stock returns and interest rates is high. However, the parameter value having the largest effect on option prices is σ_1, that is the one related to non interest rate risk, while the one that has the least effect is δ.

The sensitivity of the Jarrow formula to a change in the time-to-maturity of the option is

$$(18) \quad \frac{\partial C_J}{\partial T} \equiv \phi_J = \frac{1}{2} SN'(h) \frac{[\sigma_1^2 + (\sigma_2 + \delta T)^2]}{\xi_J} + Kf(0,T)\,P(0,T)\,N(h - \xi_J) \quad > 0,$$

and the one of the Black & Scholes model is

$$(19) \quad \frac{\partial C_{BS}}{\partial T} \equiv \phi_{BS} = \frac{1}{2} SN'(h) \frac{\sigma}{\sqrt{T}} + Kr\exp\{-rT\}\,N(h - \xi_{BS}) \quad > 0.$$

		Jarrow model						Black & Scholes model				
		T=.25						T=.25				
		σ2						σ2				
		-0.10	-0.05	0.00	0.05	0.10		-0.10	-0.05	0.00	0.05	0.10
	400	54.78	53.08	52.70	53.57	55.87	400	55.29	53.29	52.67	53.29	55.29
	450	69.08	66.66	66.29	68.02	71.66	450	70.29	67.25	66.20	67.25	70.29
K	500	69.04	66.71	66.49	68.40	72.17	K 500	70.51	67.44	66.37	67.44	70.51
	550	66.07	63.38	62.85	64.57	68.43	550	67.18	63.90	62.77	63.90	67.18
	600	55.13	53.24	52.81	53.76	56.34	600	55.69	53.47	52.78	53.47	55.69
		T=.75						T=.75				
		σ2						σ2				
		-0.10	-0.05	0.00	0.05	0.10		-0.10	-0.05	0.00	0.05	0.10
	400	52.47	52.07	52.59	54.00	56.21	400	53.97	52.64	52.19	52.64	53.97
	450	49.19	48.74	49.55	51.47	54.28	450	51.27	49.55	48.95	49.55	51.27
K	500	49.46	49.30	50.13	51.82	54.20	K 500	51.38	50.03	49.57	50.03	51.38
	550	52.46	52.11	52.67	54.15	56.45	550	54.06	52.71	52.25	52.71	54.06
	600	51.49	50.87	51.15	52.38	54.61	600	52.73	51.33	50.87	51.33	52.73

Table 4. Partial derivatives of the option values wrt the time to maturity.

Both derivatives have the same sign but can, of course, differ in magnitude, depending on the parameter values. These expressions are useful in measuring the mispricing in expression (13).

Example 4:

Table 4 gives the thetas corresponding to the parameter values of Example 1. The differences are rather small. Note that with these parameter values negative σ_2 values imply that the ξ_J is concave in time to maturity (see section 3.2.) and hence the theta smaller than the corresponding Black & Scholes theta. The opposite is valid for the positive σ_2 values.

5. *Summary*

This paper analyzed the effect of stochastic interest rates on prices of European stock options. The analyses were based on a specific parameterization of the Merton (1973) model, derived by Jarrow (1988). It was compared to the Black & Scholes (1973) model. Conditions for situations in which Black & Scholes model under- or overprices options in a stochastic interest rate economy were derived. These conditions, which should be useful in implementing the models, were shown to be a function of the interest rate volatility, the covariance between stock and bond returns and the time to maturity of the option. Examples illustrated that even for short-term options the pricing error can be significant, well above 5 %, if the correlation between stock returns and interest rates is over –.4.

References

Black, F., and M. Scholes (1973). »The Pricing of Options and Corporate Liabilities.» *Journal of Political Economy*, 81, May-June, 637–659.

Cheng, S.T. (1991). »On the Feasibility of Arbitrage-Based Option Pricing when Stochastic Bond Price Processes Are Involved.» *Journal of Economic Theory*, 53, 185–198.

Jarrow, R. (1988). »Option Valuation of Risky Assets in a Stochastic Interest Rate Economy.» Unpublished, Cornell University.

Karatzas, I., and S.E. Schreve (1988). *Brownian Motion and Stochastic Calculus.* New York: Springer-Verlag.

Kishimoto, N. (1989). »Pricing Contingent Claims under Interest Rate and Asset Price Risk.» *Journal of Finance*, 44, July, 571–590.

Merton, R.C. (1973). »Theory of Rational Option Pricing.» *Bell Journal of Economics and Management Science*, 4, Spring, 141–83.

Rindell, K. (1991). »A Test of an Interest Rate Contingent Claims Market.» Unpublished, Swedish School of Economics.

Turnbull, S.M., and F. Milne (1991). »A Simple Approach to Interest-Rate Option Pricing.» *The Review of Financial Studies*, 4, 87–120.