RATIONAL EXPECTATIONS EQUILIBRIA WITH KEYNESIAN PROPERTIES

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1. Overview

Ever since most of us can remember, it has been possible — and even customary — to categorize studies of macroeconomic behavior as «neoclassical» or «Keynesian». Here I take the term «neoclassical» to mean the equilibrium business cycle approach pioneered by Lucas (1972), Sargent and Wallace (1975), and others. By «Keynesian» I mean both the fix-price, quantity-rationing construct popularized by Malinvaud (1977), and (with some poetic license) the macroeconomic contracting models of Fischer (1977), Taylor (1980) and Okun (1981) as well.

Much has been written about the merits of each approach: their internal logic, explanatory range and policy implications have been analyzed extensively, if not conclusively. It is fair to say that the two approaches are commonly thought to be separated by a chasm and to predict diametrically opposed patterns of short-run economic behavior in response to aggregate disturbances. Keynesians focus on quantity adjustment in the short run, neoclassicals on price adjustment.1

Whatever the differences, it is well understood that Keynesian theories have little to say about the sources of nominal rigidity in prices and wages and, therefore, about wage and price change, a topic of central importance in macroeconomics. It is equally clear from the sunspots literature (see Shiller (1978), Azariadis (1981), Cass and Shell (1983)) that equilibrium theories of the business cycle focus on only one of a great number of possible rational expectations equilibria — often the stochastic analog of the golden rule. Other equilibria are typically neglected.

The main point of this paper is that at least one of the rational expectations equilibria neglected in the equilibrium business cycle approach possesses strong Keynesian properties: prices are predetermined endogenously, and quantities do all the short-run adjusting to aggregate demand disturbances. At the other extreme, the same economy typically will admit another equilibrium with strongly classical properties: quantities are predetermined endogenously.

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1 It would be fairer, but less dramatic, to state that Keynesian macroeconomics rules out short-run price adjustment while neoclassical macroeconomics is concerned with an uninteresting form of it.
dogenously, and the immediate impact of aggregate demand fluctuations is limited to price changes.

Between these two polar cases, there is a continuum of «mongrel» equilibria with intermediate features. Interestingly enough, all of these equilibria seem to behave somewhat similarly in the long run.

Whenever many rational expectations equilibria are possible in an economy, the inevitable uniqueness question arises: How do individuals agree on a consistent system of beliefs? Which one do they pick out of the possible infinitely many such systems? The question is serious for it underscores the fundamental underdeterminacy of rational expectations equilibria when securities markets are incomplete.

One instrument that may validate a particular set of expectations, and thus help anchor equilibrium, is informed central bank policy. For instance, if (and this is a big if) the central bank has the knowledge and the ability to dampen or eliminate short-term price fluctuations, an equilibrium with predetermined prices is validated to the exclusion of all others; if, on the other hand, the central bank can shield quantities from temporary aggregate disturbances, it validates an equilibrium with predetermined quantities.

The sequel is an example of an economy (Section 2) that admits both «classical» and «Keynesian» equilibria (Sections 3 and 4), followed by a discussion of the central bank policy that makes each of them «unique» (Section 5). Conclusions and extensions take up Section 6.

2. The model

We define here rational expectations equilibria in a pure-exchange, overlapping generations economy that starts at time zero and goes on forever. The beginning of each time period \( t = 0, 1, \ldots \), coincides with the appearance of a generation, labeled «generation \( \gamma \)», that consists of \( N \) identical individuals, all of whom survive for two full periods: «youth» and «old age». Individuals are endowed with \( c_i > 0 \) units of a single perishable commodity in youth, and with \( c_i > 0 \) of it in old age; they may choose, if they wish, to consume less that \( e_i \) in youth, purchase a paper asset called «money» and use their assets in old age to finance consumption in excess of old-age endowment. The endowment vector is deterministic.

All individuals are risk-averse, maximizing the expected value of the utility function \( v = u(c_t) + \beta u(c_{t+1}) \), where \((c_t, c_{t+1})\) is the consumption vector of a generation-\( t \) individual; \( u \) is monotone, differentiable and concave; and \( \beta > 0 \) is a constant.

The «money» in question is best thought of as currency, i.e., as non-interest bearing liability of the government. In order that this liability be held willingly by households, we assume that our economy is of the «Samuelson» type, that is, every young person desires to hold money when its price is constant. Mathematically, we assume that

\[ u'(e_i) < \beta u'(e_2) \]

where primes denote derivatives.

Currency is printed by the government to finance purchases of goods from private sellers. Real per-capita government purchases are an independent, identically distributed random variable, \( \bar{g}_t \), with expected value \( \mu > 0 \) and a probability measure defined over the one-dimensional Euclidean subspace \( \Gamma = \{ g | a \leq g \leq b \} \), where \( a > 0 \). We assume \( b < e_i \), i.e., government purchases never exceed the endowment of the young, which provides an upper bound on saving.

If \( M_t \) is the nominal stock of money in period \( t \) and \( m_t = M_t / p_t \) is real balances, i.e., the purchasing power of currency expressed in commodity terms, then the government budget constraint can be written in either of the following two forms

\[ p g_t = M_t - M_{t-1} \]
\[ p_{t-1} / p_t = (m_t - g_t) / m_{t-1} \]

All traders, including the government, are price takers. Commodities purchased by the government are used up with no effect on any household's marginal rate of substitution between private goods.

Public information in this economy is the history \( H_t = (g_0, g_1, \ldots, g_t) \) of realized government consumption from the beginning of time to date. The vector \( H_t \) is an element of the product space \( G_t = G \times G \times \ldots \times G \).

A rational expectations equilibrium is then a complete description of all prices, net trades
and consumption that correspond to each possible history. More formally, we have

**Definition 1:**
A rational expectations equilibrium is a sequence of functions \( (p_t, m_t, c_t^1, c_t^2)_{t=0}^\infty \), each of them mapping \( G^{t+1} \) into \( \mathbb{R} \), such that, for every \( t = 0, 1, \ldots, \) ad inf., and \( H_t \in G^{t+1} : 

(i) The vector \( (c_t^1, c_t^2) \) maximizes the expected utility of the household subject to the budget constraint \( p_t(e_1 - c_t^1) + p_{t+1}(e_2 - c_t^2) \geq 0 \) given the price \( p_t \), the function \( p_{t+1} : G^{t+2} \rightarrow \mathbb{R} \), and the history \( H_t \).

(ii) \( c_t^1 + c_t^{2-1} + g_t = e_t + e_2 + e_3 \)

(iii) \( p_t/p_{t+1} = (m_{t+1} - g_{t+1})/m_t \)

(iv) \( c_t^1 = m_t \).

The meaning of this definition is straightforward: parts (i), (ii) and (iii) specify respectively individual choice, the national income accounting identity, and the central bank’s budget identity. Part (iv) describes equilibrium in the goods (or asset) market by requiring the excess supply of goods (or demand for real balances) by the young generation to equal the combined excess demand for goods (or supply of real balances) by the old generation and the government taken together.

Real balances, \( m_t \), is clearly the important endogenous (or state) variable here. Anybody who knows the sequence of functions \( m_t : G^{t+1} \rightarrow \mathbb{R}_+ \) may reconstruct with ease the remaining functions \( (p_t, c_t^1, c_t^2)_{t=0}^\infty \) for any history \( H_t \). In particular, part (ii) of Definition 1 tells us how to construct \( c_t^1 \); part (iii) then specifies how to build \( c_t^2 \); and part (iii) defines the functions \( p_t \), recursively from \( p_{t+1} \) and \( m_t \).

Exactly how do real balances evolve over time? The answer comes from the first-order condition to the consumer problem, viz.,

\[
u'(c_t^1) = \beta E_t[(p_t/p_{t+1})u'(c_t^2) | H_t].\]

If one combines this expression with Definition 1, one obtains

\[
u'(e_t - m_t) = \beta E_t[(m_{t+1} - g_{t+1})/m_t u'(e_2 + m_{t+1} - g_{t+1}) | H_t].
\]

Every rational expectations equilibrium obviously must satisfy the stochastic difference equation (3). Conversely, any solution of (3) with the property that

\[
g_t < m_t \text{ for all } (t,H_t)
\]
is a rational expectations equilibrium, because it corresponds to the young generation saving enough to finance the largest possible value of public purchases.

As an example, we consider the case \( u(c) = c \), i.e., of risk neutrality combined with perfect substitutability. Then inequality (1) and the difference equation (3) reduce to

\[(1') \quad \beta > 1 \]

\[(3') \quad E_t(m_{t+1} | H_t) = \mu + m_t/\beta \]

In this special case, every interior rational expectations equilibrium will satisfy (3'); there will also be a number of corner equilibria, corresponding to individual decisions of the form \( c_t^1 = 0 \) and \( c_t^2 = e_t \), which violate (3').

### 3. Classical equilibria

We focus first on equilibria with flexible prices and predetermined quantities. These equilibria allow current events to be fully reflected in contemporaneous prices but not in the flow of contemporaneous net trades. If classical equilibria exist, they satisfy inequality (4) plus the deterministic analog of the difference equation (3), viz.,

\[
u'(e_t - m_t) = \beta E_t[(m_{t+1} - g_{t+1})/m_t u'(e_2 + m_{t+1} - g_{t+1}) | H_t].
\]

where the expectation is taken over the random variable \( g_t \) alone. This is so because real balances here are independent of the entire history of public consumption.

Suppose now that the offer curve of the representative household is monotone or, what amounts to the same thing, saving is an increasing function of the rate of return. Mathematically, we are assuming that, for all \( c > 0 \),

\[
u'(c) + cu''(c) > 0.
\]

This, in turn, implies that \( xu'(x - a) \) is increasing in \( x \) for all \( a, x > 0 \).
A moment's additional thought will now convince the reader that equation (5) admits a solution of the form \( m_t = F(m_{t+1}) \), where the function \( F \) is increasing and satisfies the inequality

\[
\text{s}((m_{t+1} - b)/m_t) < m_t < \text{s}((m_{t+1} - a)/m_t).
\]

Here

\[
s(R) = a \ r \ g \ m \ a \ x \ u(e_t - z, e_t + Rz)\tag{5}
\]

is the saving function of the representative household\(^2\) expressed in terms of the gross rate of return, \( R \).

Equations of the form \( m_t = \text{s}((m_{t+1} - g)/m_t) \) are the standard description\(^3\) of dynamical competitive perfect-foresight equilibria in overlapping generations models of inflation finance, when \( g \) is real government consumption. Such equilibria are known to exist whenever government purchases are »not too large«: as Figure 1 shows, there are typically two stationary values (\( m^1 \) and \( m^2 \)) of real balances corresponding to each \( g \), plus a continuum of dynamical solutions indexed on the beginning-of-time value, \( m_0 \).

If the value of \( g \) were to rise, then the graph of the difference equation would shift upward by a vertical distance everywhere equal to the change in government consumption.

A »classical« equilibrium is a solution of equation (5) that satisfies \( m_t > b \) for all \( t \). Therefore, if government consumption is »not too large« on average, there are two stationary classical equilibria, which correspond to intersections of the \( 45^\circ \) line with the frontier labeled (C) in Fig. 1.

For each classical equilibrium — stationary or dynamical — net trades and youthful consumption are completely predetermined by initial conditions, preferences, endowments and the probability distribution of public spending.\(^5\) Actual government purchases affect the price level (see Definition 1) and old-age consumption, which they displace one-for-one. Thus, the main allocative effect of unexpected government expenditure in a classical equilibrium is completely to crowd out consumption by holders of money balances.

4. Keynesian equilibria

At the opposite extreme from »classical« equilibria, we seek solutions to equation (3) that are associated with predetermined prices and flexible net trades. It is worth emphasizing here that price inflexibility neither assumes nor implies quantity rationing. If Keynesian equilibria of this type exist, they correspond to solutions of the form

\[
m_t = \text{s}((m_{t+1} - g_{t+1})/m_t)\tag{6}
\]

to the stochastic difference equation (3).

Figure 1 indicates again the Keynesian equilibria exist if \( b \) is »not too large«: equation (8) describes a family of two-dimensional curves, one for each possible value of \( g \), in the interval \([a, b]\). The extreme members of this family, labeled \( \text{K(a)} \) and \( \text{K(b)} \), correspond to the two extreme values of government spending. These are shown in Figure 1.

Stationary Keynesian equilibria do not exist in the usual, deterministic sense. However, it is simply means that the economy is of the »Samuelson« type.

\(^2\) Non-negative saving means individual liabilities do not circulate as stores of value, only fiat money does. This is an extreme form of a liquidity constraint.

\(^3\) See Phelps (1973), for example.

\(^4\) The frontiers drawn in Fig. 1 are reflected offer curves, which connect individually optimal choices of youthful excess supply for goods with old-age excess demand, at different values of public consumption. All these offer curves have slope less than unity at \( m_t = 0 \), which

\(^5\) This feature of »classical« equilibria has already been noted in Farmer and Woodford (1984), and Azarias and Cooper (1983).
clear from Figure 1 that $m_t$ will tend to stay in the interval $[m'(a), m'(b)]$ if it begins there. Following Farmer and Woodford (1984), one may exploit this thought to show that Keynesian equilibria will tend asymptotically towards a stationary probability distribution on the interval $[m'(a),m'(b)]$.

Unlike the »classical» case of the previous section, Keynesian equilibria have a strong impact on net trades and consumption by the young, but no impact whatsoever on contemporaneous prices or consumption by the older generation. Since government spending does not affect the short-run rate of inflation in a Keynesian equilibrium, the usual inflation tax works with a lag: public consumption displaces the private consumption of future money holders, not of current ones. Crowding out thus smites the young generation rather than the old one.

A good example of the differences between the two polar types of equilibrium is the perfect-substitutes case in equation (3). Here interior »classical» equilibria are associated with solutions of

$$(8a) \quad m_{t+1} = \mu_{t+1} + m_t / \beta$$

in the interval $[b, e_i]$; interior »Keynesian» equilibria correspond to solutions of

$$(8b) \quad m_{t+1} = g_{t+1} + m_t / \beta$$

in the interval $[0, e_i]$.

The geometry of Figure 2 shows that equilibria of the former type exist if

$$b < \beta \mu / (\beta - 1) < e_i;$$

all of these converge monotonically to $\beta \mu / (\beta - 1)$. For Keynesian equilibria, on the other hand, equation (8b) says that:

(i) The rate of price inflation is independent of history, and in particular, $p_{t+1} = \beta p_t$, all $t$.

(ii) The dynamical path of real money balances,

$$(10a) \quad m_t = \beta^{-1} m_0 + \sum_{s=1}^t \beta^{-s} g_{t-s}$$

stays in the interval $[a \beta / (\beta - 1), \beta \beta / (\beta - 1)]$ if it begins there, i.e., if $m_0$ lies in that interval. Therefore, an equilibrium with predetermined prices exists if

$$(10b) \quad b \beta / (\beta - 1) \leq e_i.$$

5. Central bank policy

Both »classical» and »Keynesian» equilibria are genuine rational expectations equilibria of same economy, and yet they differ substantially in their qualitative properties, especially in what they predict to be the consequences of aggregate demand shocks. The situation is further complicated when one realizes that, typically, there is a continuum of equilibria.

To understand why the set of equilibrium allocations is so large, we return to the stochastic difference equation (3), and suppose that we have found two distinct solutions of it,

$$(11) \quad m_{t+1} = F^1(g_{t+1}, m_t) \quad m_{t+1} = F^2(g_{t+1}, m_t).$$

Here we assume that $(F^1,F^2)$ are functions of the form $F^i : [a,b] \times [b,e_i] - [b,e_i]$ for $i = 1, 2$. Then we can construct another function, $F^0$ such that $m_{t+1} = F^0 (g_{t+1}, m_t)$ also is a rational expectations equilibrium. To that end, we define $F^0$ from

$$(12) \quad (F^0-g_{t+1}) u'(e_2 + F^0-g_{t+1})$$

$$\quad = (F^1-g_{t+1}) u'(e_2 + F^1-g_{t+1})$$

$$\quad + (1-\theta) (F^2-g_{t+1}) u'(e_2 + F^2-g_{t+1})$$

for an arbitrary $\theta \in (0, 1)$. 

Figure 2.
Next we observe that $F^0$ lies between $F^1$ and $F^2$ for all $(g, \tau, m)$, provided that the offer curve is monotone (i.e., $xu'(x)$ is an increasing function of $x$). Therefore, $F^0$ maps $[a, b] \times [0, e]$ into $[b, e]$ and satisfies equation (3) by definition for all $\tau \in [0, 1]$. That makes $F^0$ a rational expectations equilibrium for each $\tau$, and implies that there is a continuum of such equilibria.

The specter of a continuum of rational expectations equilibria with dissimilar qualitative properties is not a pleasant one to contemplate. How does a particular equilibrium arise in this economy, anyway? One way to think about this issue is to put one's faith in the Coase theorem, and exclude all equilibria that are pareto-inferior on the grounds that the economy will gravitate somehow towards undominated equilibria.

Another, somewhat more attractive, device to reduce the indeterminacy of equilibrium is to realize that, as the monopoly supplier of currency, the central bank does not take prices as given, even if government purchases are made competitively. A central-bank policy is a sequence of rules $p_t: \Delta^t \rightarrow R$ describing how much money the bank creates each period as a function of the history of government expenditure up to that point.

Equilibrium may not exist for some or many policy choices. For instance, if the central bank created each period a quantity of money that is sufficient to finance government purchases only at some constant price level, say $\bar{p}$, then competitive equilibrium will generally fail to exist. A happier outcome is possible if policy rules are chosen with some care and, one must admit, a good deal of knowledge about the structure of the economy.

This syllogism is best demonstrated with a concrete example. We return to perfect substitutes and equation (3)' which, we saw in the previous section, admits a Keynesian equilibrium of the form $m_{t+1} = g_{t+1} + m_t/\beta$, with prices $p_t = p_0 \beta^t$ and $\beta > 1$. If the central bank wishes to validate this particular equilibrium to the exclusion of all others, then its policy rule has to specify that money be printed to finance government purchases at a constant rate of inflation equal to $\beta - 1$. Specifically, money supply must follow the rule

$$M_t = M_0 + \sum_{i=1}^{t} p_i g_i = M_0 + p_0 \sum_{i=1}^{t} \beta^i g_i$$

which assigns greater influence on the current stock of money to more recent government purchases.

Similar considerations apply to classical equilibria, and the argument easily extends outside the perfect-substitutes case. In general, we suppose that, for $t = 1, \ldots, \infty$, a sequence of functions $p_t^*: \Delta^t \rightarrow R$ is associated with a particular equilibrium; then a central bank that knows this sequence and controls with pinpoint accuracy the stock of money can implement this equilibrium by following the policy

$$M_t = M_0 + \sum_{i=1}^{t} p_i^*(H_t) g_i.$$

6. Conclusions

The central point of this paper was to construct a fairly robust example of an economy that admits a continuum of rational expectations equilibria without quantity rationing. Some of these equilibria differ substantially in what they predict to be the consequences of an aggregate demand shock. One of them has pronounced «classical» features: flexible prices; public consumption crowding out the private consumption of the older generation; and a completely predetermined sequence of equilibrium net trades (i.e., real balances) that reflects the probability structure of demand shocks but not their realized values.

Another equilibrium has strong Keynesian properties: no individual is rationed but prices are nevertheless predetermined one period in advance; public consumption crowds out private consumption by the young generation; and net trades reflect the entire history of government purchases.

These equilibria survive a number of natural extensions to the economy at hand such as finitely many physical commodities (rather than just one), finitely many types of households per generation, production in addition to pure exchange, taxes, random endowments, etc. All that seems to be required for the existence of these equilibria is some upper bound on public consumption.

Here, as in many dynamical monetary economies with one paper asset (or with many perfectly substitutable ones), the central bank cannot affect the set of equilibrium allocations, but it seems to have considerable in-
fluence on the actual choice of equilibrium from within that set. Given sufficient knowledge of the economic structure and sufficiently precise control of the stock of money, the central bank »guides» the economy to any equilibrium allocation simply by reacting to events in a particular way, that is, by picking the appropriate »policy rule».

What if the central bank has insufficient understanding of the economy and/or inaccurate control of money supply? Equation (14) says that the central bank must know the (sequence of) price functions that correspond to the equilibrium it wishes to implement, just as households must if the expectations are to be formed rationally.

Control of money supply is conceptually a different issue. We saw how error-free policy rules can guide the economy to any desired point in the set of equilibrium allocations. If the setting of the money stock involves independent, non-cumulating errors, then a reasonable conjecture to investigate would be that there may exist a policy rule that possesses a sufficiently high probability of placing the economy in the »neighborhood» of the »desired» equilibrium.

What equilibria the central bank may desire is a question worth pondering. Assuming central bankers have at heart the welfare of the »representative» household, they might inquire as to which of the infinitely many rational expectations equilibria best serve that welfare. It is pretty obvious, for instance, that in the perfect-substitutes, risk-neutrality case of equation (3)', the expected utility of each generation t depends neither on past history nor on the actual equilibrium the economy happens to be in. Whether more general economies admit a partial ordering of equilibria that is useful for central bank policy remains to be seen.

References

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