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GROWTH, INFLATION, AND ECONOMIC POLICY IN A STOCHASTIC CASH-IN-ADVANCE ECONOMY

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We develop a continuous-time stochastic growth model with recursive preferences, money and public debt. In equilibrium growth and inflation follow geometric Brownian motions, with parameters determined by solving a system of nonlinear equations. Permanent changes in government expenditures and taxes have both real and nominal effects producing often reverse Mundell–Tobin effects. Superneutrality holds when money supply changes are caused by open market operations, irrespective of the primary fiscal stance. The magnitude of the policy effects are examined using a calibrated version of the model. (JEL E63, H63)

1. Introduction

The effects of fiscal and monetary policies on economic growth and inflation continue to be a major concern for macroeconomic analysis. An important issue is the incorporation of random shocks and uncertainty to the discussion of the impacts of economic policies.[1] With random shocks the risk and intertemporal substitution attitudes of the representative consumer can exert a significant influence on the rate of growth through aggregate saving and investment behaviour. They also affect the magnitudes of policy effects on growth and inflation.

Under uncertainty it is important to distinguish between intertemporal substitution and risk aversion. However, most of the previous literature has failed to do this since intertemporally separable isoelastic or even logarithmic preferences have been assumed. In contrast we will postulate so-called recursive non-expected utility preferences in which separate parameters describe the degrees of risk aversion and intertemporal substitution.

In the earlier literature Eaton (1981) and Corsetti (1992) assumed separable isoelastic preferences and studied normative implications of fiscal measures in a simple nonmonetary growth model with government deficits. Smith

[1] *Early papers include Eaton (1981), Gertler and Grinols (1983), Aiyagari and Gertler (1985), and Danthine, Donaldson, and Smith (1987). Turnovsky (1993) contains references to the more recent literature.*

(1995) allows for recursive preferences in a similar framework with balanced budgets.[2] Monetary growth models with uncertainty have been based on either the assumption of risk neutrality (see Aiyagari and Gertler 1985) or logarithmic preferences (Grinols and Turnovsky 1993, Turnovsky 1993 and Ireland 1994). In this paper we reconsider the effects of fiscal and monetary policies on growth and inflation emphasizing the consequences of modes of behaviour under uncertainty in a dynamic framework.

We develop a simple continuous-time stochastic model of growth and government finance. The fundamental source of uncertainty is taken to be productivity shocks in a linear stochastic production technology. The intertemporal consumption-portfolio decisions of the consumer are modelled assuming recursive non-expected utility preferences and the cash-in-advance constraint. The consumer faces asset returns which are geometric Brownian motions derived, in general equilibrium, from the fundamental productivity shocks. (The latter are assumed to follow a geometric Brownian motion.) In modelling the production sector a linear production function of capital is assumed, so we follow the earlier literature in this respect (compare e.g. Eaton 1981, Turnovsky 1993 and Smith 1996). The model of the economy is completed by incorporating a government purchasing a fraction of output and financing it through income taxation and issues of bonds and money.

Though the model is highly stylized, it is appealing as it permits an analytic solution. (The corresponding discrete-time system can only be solved using quadratic-linear or other approximations.) The solution is shown to be unique and feasible for a wide range of parameter values. The relevant variables are shown to follow Brownian motions whose mean and stochastic components are determined by a system of nonlinear algebraic equations. The system is nonlinear but well-behaved. Our discussion of the effects of policy on equilibrium growth relies on both qualitative and quantitative results. The latter are based on a calibration of the model and the numerical solution of the equations characterising the equilibrium.

Our calibrated model permits a quantitative comparison of fiscal policy consequences in mildly and strongly risk-averse economies. Crowding-out of growth by increased government expenditures is stronger in the more risk-averse economy. In contrast, the real effects of tax cuts are at their strongest in relatively risk-neutral economies and can even reverse their sign with sufficient risk aversion. Combining these with the effects on inflation leads to the possibility of reverse Mundell-Tobin effects.

Money supply changes caused by open market operations are superneutral irrespective of the primary fiscal stance. This corresponds to the finding of Danthine, Donaldson, and Smith (1987), who concluded for Sidrauski-type models with uncertainty that superneutrality of money for »helicopter drops» depends on the modelling details, but that in any case the degree of nonneutrality is very small.

The paper is organized as follows. After setting out the basic framework in Section 2 we briefly take up two special cases of the model: (i) no money and public debt and (ii) nonmonetary economy with public debt and deficits. They facilitate comparison to the monetary model. The complete model and its solution are described in Sections 4 and 5. Section 6 discusses the calibration for the discussion of growth, inflation and policies which is done in sections 7 and 8, respectively.

2. The Basic Framework

We will study growing economies with flexible prices and operating in continuous time. Uncertainty is present through aggregate supply shocks. As discussed in the introduction, the models are based on the notion of an infinitely-lived representative consumer who has recursive non-expected utility preferences over infinite consumption streams. The intertemporal objective function U(t) is defined by the limit $h \to 0$ of the recursion

[2] *Obstfeld (1994a, b) utilizes recursive preferences in studies of consumption variability and opening of international capital markets.*

$$(1)\quad F[(1-R)U(t)] = \frac{1-R}{z} C^z h + e^{-vh} F\,[(1-R)E_t U(t+h)],$$

where $F(x) = \frac{1-R}{z} x^{z/(1-R)}$. Here the parameter R measures relative risk aversion and $(1-z)^{-1}$ is the elasticity of intertemporal substitution. This parametric specification was introduced by Svensson (1989), and it has recently been utilized by e.g. Obstfeld (1994a, b) and Smith (1996).[3]

We assume for convenience that firms use a simple linear technology, in which real capital k is the only input.[4] Output flow dY is uncertain and follows an Ito process. We interpret the source of uncertainty as random shocks in the productivity of capital in a linear production function. Formally, we specify the flow of output as follows:

$$(2)\quad dY = (rdt + \sigma dz)k,$$

where $(dz)^2 = dt$. This formulation is a special case of the general model of Cox, Ingersoll and Ross (1985). (2) has also been used by Eaton (1981), Gertler and Grinols (1983), Honkapohja and Lempinen (1990), Corsetti (1992), Turnovsky (1993), Obstfeld (1994b) and others.

The stock of capital is owned by a single infinitely-lived household, from whom the firms hire capital. The rental of capital is paid continuously over the period of the lease. Thus in real terms the zero-profit condition of linear technologies implies that households receive an income flow equal to dY as rental at each moment of time.

The consumer makes decisions about his/her consumption and asset portfolio. The nature of the wealth constraint depends on the availability of different types of assets. In the most general model there will be three assets: real capital, (nominal) government bonds and money, with the demand for money based on the cash-in-advance constraint.

The government is assumed to follow exogenous expenditure and income tax rules. More precisely, the government purchases a fixed fraction g of output produced in the economy and collects a fraction s of rental and interest income as taxes. In the most general model the government can also partly finance its purchases by issues of bonds and money.[5]

Since our general model is quite complex, we will develop the analysis in stages by first considering briefly two simplified models. This will also facilitate comparison to other literature. In the first special case government purchases are fully financed by the income tax, and there do not exist any government bonds. In the second special case the government can also issue real debt and partly finance its operations through that means, but there is no money in the economy.

3. Special Cases

3.1 A Model without Government Debt

In the first special case we assume that the government primary revenues and spending are balanced, and there are no government bonds or money as assets in the economy.[6] Consumer wealth consists of real capital k and it evolves in accordance with

$$(3)\quad dk = dY - Cdt - dT,$$

where dT denotes the income tax of the consumer. It is given by

$$dT = sdY,$$

while government expenditure is assumed to be the corresponding proportion of the output:

$$dG = gdY, \text{ where } g = s.$$

[3] *See Duffie and Epstein (1992) for a full treatment of the theoretical underpinnings of recursive utility in continuous-time stochastic situations.*

[4] *This kind of production function is often used as a simplification. In the literature on endogenous growth it was introduced by Rebelo (1991).*

[5] *The model is thus related to the analyses of Calvo (1985), McCallum (1984), Hartman (1987, 1988), Grinols and Turnovsky (1993) and Turnovsky (1993).*

[6] *See Smith (1996) for a similar framework.*

Changes in consumer's wealth thus take the form

(4) $dk = (1-s)(rdt + \sigma dz)k - Cdt,$

and his optimization problem is to maximize (1) subject to (4) and a solvency condition. As is well-known, the solution is given by the consumption function

(5) $C = (1-z)^{-1}[v - z(1-s)r + 2^{-1}Rz(1-s)^2\sigma^2]k \equiv Vk.$

Combining this with the market-clearing condition

(6) $dk = dY - Cdt - dG$

and government budget balance (s = g) yields for the growth rate of the economy the formula

(7) $dk/k = [(1-g)r - V]dt + (1-g)\sigma dz = (1-z)^{-1}[(1-s)r - v - 2^{-1}zR(1-s)^2\sigma^2]dt + (1-s)\sigma dz.$

We can draw some useful conclusions from equation (7). First, the importance of risk aversion is seen from the expected growth rate $(1-z)^{-1}[(1-s)r - v - 2^{-1}zR(1-s)^2\sigma^2]$. The effect of uncertainty on growth depends on the degrees of risk aversion and intertemporal substitution. If the latter is inelastic we have $z < 0$, and in such economies a higher value of risk aversion R or production volatility σ^2 implies more rapid growth.[7] The relation between mean growth rate and intertemporal substitution parameter z is theoretically ambiguous: with parameter values of Section 6 and small values of R the relation is increasing in z, but with R high enough growth is decreasing in z. This is in contrast to the case of certainty for which the growth rate is always increasing in z.

Second, the strength of the real effects of balanced-budget expenditure or tax changes depends on the magnitudes of production volatility, risk aversion and intertemporal substitution in the economy. The derivative of mean growth rate with respect to s is given by $(1-z)^{-1}[-r + zR\sigma^2(1-s)]$. If $z < 0$, the negative growth effects of (balanced-budget) tax increases are stronger with more risk-aversion, higher production volatility and less elastic intertemporal substitution (smaller z).

3.2 A Model with Government Debt

We now extend the model by incorporating imbalances in the primary budget financed by issues of public debt ($g \neq s$). The model still remains a real model, as money and inflation will not be considered. Our treatment will be brief and will just highlight some central points.[8]

Denoting by b government net debt in real terms its instantaneous real return is conjectured to be $r_b dt + \sigma_b dz_b$. (The equilibrium solution will be computed below). Assuming that the real return on debt is taxable income to the consumer, the government and consumer's budget constraints are, respectively,

(8) $db = dG - sdY + (1-s)b(r_b dt + \sigma_b dz_b),$

(9) $dw \equiv dk + db = dY + b(r_b dt + \sigma_b dz_b) - Cdt - dT.$

The tax rule is now $dT = s(dY + b(r_b dt + \sigma_b dz_b))$, while again the government is assumed to purchase a fixed proportion of output i.e. $dG = gdY$.

The consumer's optimization problem consists of maximizing (1) subject to (9), taking into account the tax rule. Letting $x = k/w$ the share of capital in wealth w, its solution is given by the consumption function

(10) $C = Vw \equiv (1-z)^{-1}(v - zr_w + .5zR\sigma_w^2)w$

and the optimal portfolio rule

(11) $$x^* = \frac{r - r_b}{(1-s)R\sigma_s^2} + \frac{s_b^2 - \sigma_{kb}}{\sigma_s^2}.$$

[7] *This result is reversed if 1 > z > 0 i.e. intertemporal substitution is elastic. Subsequently, the case 1 > z > 0 is omitted for reasons of brevity.*

[8] *See the early paper by Eaton (1981) for a more extensive discussion with slightly more general expenditure and tax rules but less general preferences. Corsetti (1992) is a reconsideration of Eaton's analysis. Note that we are able to provide a full solution to the model and not just a characterization of the equilibrium.*

In (10) $r_w = (1-s)[(r-r_b)x^* + r_b]$ denotes the mean return on wealth and $\sigma_w^2 = (1-s)^2[(x^*)^2\sigma^2 + 2x^*(1-x^*)\sigma_{kb} + (1-x^*)^2\sigma_b^2]$ its variance, using the optimal portfolio x^*. Here σ_{kb} and σ_b^2 are, respectively, the covariance of dz and dz_b and the variance of dz_b. In addition, we have introduced the notation $\sigma_s^2 = \sigma^2 - 2\sigma_{kb} + \sigma_b^2$. Equation (11) has the usual interpretation: the first term is the »speculative portfolio» and the second term the minimum variance portfolio.

To obtain the full solution of the model we require balanced growth, i.e. dw/w = dk/k = db/b. We first note that the market-clearing condition (6) implies that in equilibrium $r_w = (1-g)r + V(1-x^{-1})$ and $\sigma_w^2 = (1-g)^2\sigma^2$. Using (10) one obtains the equation

$$(12) \quad V = x(x-z)^{-1}[v - z(1-g)r + .5zR(1-g)^2\sigma^2] \equiv x(x-z)^{-1}A.$$

From the identity defining A we see that A depends only on exogenous parameters, including the govt expenditure parameter g.

Second, we note that the government budget constraint can be written in the form

$$db/b = (g-s)(rdt + \sigma dz)(k/b) + (1-s)(r_b dt + \sigma_b dz_b).$$

From the requirement dk/k = db/b one obtains the equations

$$(1-g)r - V/x = \frac{(g-s)x}{1-x}r + (1-s)r_b$$

$$(1-g)\sigma dz = \frac{(g-s)x}{1-x}\sigma dz + (1-s)\sigma_b dz_b.$$

Clearly, $dz = dz_b$ must hold. Combining these equations with (11) and (12) leads after tedious algebra to

$$(13) \quad x = \frac{A + Bz(g-s)}{A - B(g-s)},$$

where $B = r - R(1-g)\sigma^2$ depends only on exogenous parameters, including the govt expenditure parameter g. The solution values for other variables can easily be computed from (13). For the rate of growth we obtain

$$(14) \quad dw/w = (1-g)(rdt + \sigma dz) - (V/x)\,dt,$$

which is of the same form as (7), except for the change in the drift term caused by the fact $x \neq 1$.

We have done simple numerical experiments which show that for $g = 0.21 > s = 0.2$ (and numerical values of preference and production parameters discussed below in section 6) the growth rate is increasing in the risk aversion parameter R (for $z < 0$). Its dependence on z is in general uncertain. With the chosen parameter values growth rate is increasing in z for small values of R but decreasing when R is large enough. This result is in accordance with the simple model in Section 3.1.

(13) and (14) yield the important conclusion that the steady state equilibrium depends only on the policy parameters g (through A and B) and the primary deficit parameter $g-s$. The rate of proportional income taxation s influences the equilibrium only via this primary deficit parameter. In particular, with small deficits ($g > s$), but ($g \simeq s$) the effects of small tax changes on growth are null. The effects of changes in s can be significant only when $g-s$ is nonnull.

In contrast, the effects of changes in g on growth are always nontrivial. As we have assumed that government expenditure is pure consumption and separable from private consumption, increases in g can naturally be expected to crowd out growth. This is borne out by numerical examples (with parameter values of section 6).

4. *Equilibrium with Money, Debt and Growth*

We now come to the main body of the paper in which money and inflation are incorporated into the preceding model. In the complete model there are three assets, so that the representative consumer can save by holding stocks of real capital, bonds and/or money. Let

$$(15) \quad w \equiv k + b + m \equiv k + B/P + M/P$$

be the stock of real wealth. The consumer, who receives real income through rentals on capital and interest payments on bonds, spends it on consumption (Cdt), taxes (dT), the inflation tax (dS) or accumulates real capital (dk), stock of bonds (db) or real balances (dm). The accumulation equation of real total wealth can then be written as

$$(16)\quad dw = dk + db + dm = dY + ibdt - Cdt - dT - dS$$

in which i is the riskless nominal rate of interest paid on government bonds.

The instantaneous government expenditure flow is again

$$dG = gdY.$$

For income taxation it is now assumed that the real tax flow takes the form

$$dT = s(dY + ibdt).$$

This tax rule means proportional taxation of nominal income.

The government finances budget deviations by issues and withdrawals of money and nominal bonds. Given the expenditure and tax rules and the bond-issuing policy, financing the government budget deviations requires that money supply must follow the stochastic process

$$(17)\quad dM/M = (dG - dT)/m + (idt - dB/B)(b/m).$$

The inflation tax comprises two components, as the consumer is subject to capital losses caused by inflation through both bond and money holdings. Let the candidate stochastic process for the inflation rate be[9]

$$dP/P = \pi dt + \sigma_p dz_p.$$

Then we have for the inflation tax the result

$$dS = dS_1 + dS_2 = [dP/P + (dP/P)(dB/B) - (dP/P)^2]b + [dP/P + (dP/P)(dM/M) - (dP/P)^2]m.$$

[9] *Later the processes dB/B and dP/P will be determined as part of the general equilibrium. Note that govt budget deficits or surpluses will influence the covariance between the returns to capital and bonds.*

Using the definitions for the productivity process, the tax rule, and adopting the notation

$$\pi_b = \pi + (dB/B)(dP/P) - (dP/P)^2$$

$$\pi_m = \pi + (dM/M)(dP/P) - (dP/P)^2$$

we can write the budget constraint of the consumer in the following explicit form

$$(18)\quad dw = (1-s)(rdt + \sigma dz)k + ((1-s)i - \pi_b)bdt - b\sigma_p dz_p - Cdt - \pi_m mdt - m\sigma_p dz_p.$$

The demand for money in the economy is determined by a cash-in-advance constraint, which is assumed to take the form

$$(19)\quad M = PC,$$

where P is the price of consumption. The binding form of the constraint is justified by the utility function (1).[10] In (4.5) the velocity of money is set at unity by an appropriate choice of the monetary unit.

The consumer maximizes (1) subject to the budget equation (18) and the cash-in-advance constraint (19). The solution for the demand for real capital k and the rate of consumption in terms of the real wealth can be shown to be (see appendix 1):

$$(20)\quad x = \frac{(1-s)(r-i)+\pi_b}{R\sigma_t^2} + \frac{\sigma_p^2 + (1-s)\sigma_{kp}}{\sigma_t^2}$$

$$(21)\quad C = (1 + (1-s)i - \pi_b + \pi_m - z)^{-1}[v - zr_w + 0.5zR\sigma_w^2]w \equiv Vw,$$

where

$$x = k/w,$$
$$\sigma_t^2 = (1-s)^2\sigma^2 + \sigma_p^2 + 2(1-s)\sigma_{kp},$$

$$(22a)\quad r_w = \{k(1-s)r + b[(1-s)i - \pi_b] - m\pi_m\}w^{-1}$$

[10] *A corresponding formulation of the cash-in-advance constraint has been used in continuous-time models by Calvo (1986, 1987) and others.*

(22b) $\sigma_w^2 = x^2\sigma_t^2 - 2x(\sigma_p^2 + (1-s)\sigma_{kp}) + \sigma_p^2.$

Here (20) is the portfolio rule, while (21) defines the propensity to consume V. They can be usefully compared to (10) and (11).

The portfolio rule (20) has the same general interpretation as (11) in terms of minimum variance and speculative components, though as a result of the form of the return to bonds and the tax structure the forms of the two components are different. Note also that the portfolio fraction $x = k/w$ includes the real money stock in the definition of w. Comparing (21) and (10) it is seen how the consumption propensity is altered. It now depends on the after-tax nominal interest factor $1 + (1-s)i$ as well as on the difference between the two inflation terms $\pi_m - \pi_b$.

The consumption rule (21) directly implies that the demand for real balances equals

(23) $m = (1 + (1-s)i - \pi_b + \pi_m - z)^{-1}$
$[v - zr_w + .5zR\sigma_w^2]w = Vw.$

Equations (20)–(23) summarize the demand and allocation behaviour of the representative consumer.

We can now formulate the concept of a general equilibrium. The definition adopted here is that of *stochastic balanced growth,* so that consumption, real capital, real stock of bonds and real balances all grow at the same stochastic rate as geometric Brownian motions, while the proportions k/w of wealth invested in real capital and government debt b/w remain constant over time.

This kind of equilibrium is achieved, if the following requirements hold:

1. The representative agent chooses optimally V,x and m;
2. Private and government budget constraints are binding;
3. $dk = dY - dG - Cdt$ (goods market equilibrium condition);
4. $M^s/P = M^d/P$ (money market equilibrium condition);
5. $m^d/b^d = m^s/b^s$ (asset market equilibrium condition);
6. m^s/b^s = constant (fixed government financing structure).

The last requirement that open-market operations be carried out so as to fix the bonds-to-money ratio is an assumption also made by Calvo (1985), Grinols and Turnovsky (1993), Turnovsky (1993) and others. Such an assumption implies (stochastic) monetary growth targeting by the central bank.

5. Solution of the Complete Model

The solution of the general model is done in two steps as follows. First, for given values of the fraction of portfolio held in real capital x, the propensity to consume V, and nominal interest rate i we can determine both the deterministic and stochastic components of the geometric Brownian motions for the rates of growth, inflation, and nominal money (and bond) growth.

Second, the asset market equilibrium condition, the bonds-money ratio and the cash-in-advance constraint give three relationships between i, x and V. Two of them can be combined to eliminate V, so one obtains a system of two nonlinear equations for i and x. This system can be analyzed using standard techniques, but it turns out that unambiguous theoretical results are scarce. Numerical solutions can be obtained, so that comparative-dynamic results are obtainable for particular values of basic model parameters. Below we will discuss both the theoretical and the numerical results in an integrated fashion.

5.1. Step I: Equilibrium Equations

We now start with the first step of the solution procedure. The assumption on open-market operations implies that the nominal stocks of money and bonds must follow the same stochastic processes. Formally, we have that

(24) $dM/M = dB/B.$

Equation (24) also implies that in this analysis the mean inflation rates π_b and π_m are identical.

Utilizing (24) in the government budget constraint, and denoting by f the bonds-money ratio ($f = b/m$), we have that

(25) $dM/M = (\alpha r + (1-s)fi(1+f)^{-1})dt + \alpha\sigma dz,$

(26) where $\alpha = \alpha(x) = x(g-s)[(1+f)V]^{-1}$.

Money (and bond) supply thus follows an Ito process, which is a transform of the basic productivity process and the nominal interest rate. Its full solution requires the equilibrium values for i, x and V.

It can readily be seen from the goods market equilibrium condition that by using the definitions for output, capital share x, the expenditure rule (21), and the general expression for the consumption rule the general equilibrium solution for the rate of growth of real wealth is again given by

(27) $dw/w = (1-g)(rdt + \sigma dz) - (V/x)dt.$

The money market equilibrium condition states that money demand and money supply must be equal. In flow terms, using the government budget constraint and the money demand equation (23), this condition can be rearranged into a condition on the rate of inflation. Specifically, the inflation rate that satisfies the money market equilibrium, is given by

$$\pi_b dt + \sigma_p dz_p = dM/M - dw/w$$

Using (25) and (27) this yields

(28) $\pi_b dt + \sigma_p dz_p = [\delta r + (1-s)fi(1+f)^{-1} + (V/x)]dt + \delta\sigma dz,$

where $\delta = \delta(x) = \alpha(x) - (1-g)$ and $\alpha(x)$ is given by (26). According to (28) inflation follows an Ito process.

The inflation rate has the intuitive interpretation that it is the difference between the rate of growth of money supply and the rate of growth of real wealth. A policy measure can affect inflation both directly through money supply effects and indirectly through growth effects, breaking in such cases the classical dichotomization.

On the other hand, the mean inflation rate depends on the nominal interest rate, the solution for which can only be obtained by jointly using the asset market equilibrium condition. The latter can be formalized as follows:

(29) $x = 1-(1+f)V.$

Denoting by x_m the minimum variance of portfolio rule (20) for x the asset market equilibrium condition requires that the mean inflation rate has to satisfy the following equation

(30) $\pi_b = (1-s)i - (1-s)r - R\sigma_t^2[x_m - x],$

with

(31) $\sigma_t^2(x_m - x) = -(g-s)(1-g)\sigma^2[(1+f)V]^{-1}.$

The last term $-R\sigma_t^2(x_m-x)$ in (30) is the asset market risk premium which drives a wedge between a Fisher parity type of relationship and the actual determination of after-tax nominal interest rate in (30). If g = s, there is no risk premium, and the Fisher-type relation holds. The form of the latter depends on the tax rate s, since we assumed taxation of nominal income.

Given the bonds-money ratio f and the consumption-saving and capital allocation decisions V and x of the representative agent, the money and asset market conditions determine jointly the nominal interest rate and the mean inflation rate compatible with the general equilibrium. Solving (28) and (30) we get for the mean inflation rate:

(32) $\pi_b = (g-s)V^{-1}r + fR\sigma_t^2(x_m - x) - [(1-s)r - (1+f)(V/x)].$

It is seen from (28) and (32) that we have a solution for the inflation rate once the equilibrium values of x and V are available. From the same computation we get for the nominal interest rate

(33) $(1-s)i = V^{-1}(g-s)r + (1+f)R\sigma_t^2(x_m - x) + (1+f)V/x$

Computing r_w and σ_w^2 of (22a–b) by means of (18), (25) and (30) we obtain

(34) $V = \mathcal{A}x(x(1+(1-s)i) - z)^{-1},$

where $\mathcal{A} = [v - z(1-g)r + 2^{-1}zR(1-g)^2\sigma^2].$

5.2. Step II: Existence and Uniquess of the Solution

The second step for determining the equilibrium consists of the solution of the equilibrium values of i, x and V from the system of three equations (29), (33) and (34). Here one needs to use (31) to substitute for $\sigma_t^2(x_m - x)$ in (33) in terms of f and V.

We show next that (for z < 0) the system of equations has a unique solution, when primary deficits are not too large. Substituting (34) into (29) and defining a new variable $I = 1 + (1-s)i$ we get

$$x^2I - xz - xI + z + (1+f)\mathcal{A}x = 0, \text{ or}$$

(35) $$I = (x\psi - z)/(x^2 - x) \equiv h(x;\psi),$$

where $\psi \equiv z - (1+f)\mathcal{A} < 0$. It is easy to check that h(x) is continuous and $h'(x) > 0$ for $x \in (0,1)$. The limits at the endpoints are $h(x) \to -\infty$, as $x \to 0+$, and $h(x) \to +\infty$, as $x \to 1-$, so that h(x) obtains all real values in this domain.

Using the formula (31) for the risk-premium and substituting (34) into equation (33) we get

(36) $$\mathcal{A}(xI-1) - \mathcal{B}(xI-z) = 0,$$

where $\mathcal{B} \equiv (g-s)[r - R\sigma^2(1-g)]$. (36) leads to the equation

(37) $$I = (\mathcal{A} - \mathcal{B}z)/[(\mathcal{A}-\mathcal{B})x] \equiv j(x;\mathcal{A},\mathcal{B}).$$

Since usually $\mathcal{A} > \mathcal{B}$, it is easy to see that $j(1) > 1$, $j'(I) < 0$ for $x < 1$, and $\lim_{x \to 0+} j(x) = +\infty$. Combining the properties of (35) and (37) one obtains the basic result:

Proposition 1: A unique steady state equilibrium exists in the economically relevant domain $0 < x < 1$, $I > 1$ when the primary budget is in balance or the deficit/surplus is not too big.

Once the solution values for i, x and V are obtained, the equilibrium rates of growth and inflation can be fully determined, and they follow specific geometric Brownian motions. Later we provide graphs illustrating simulations of the inflation process.

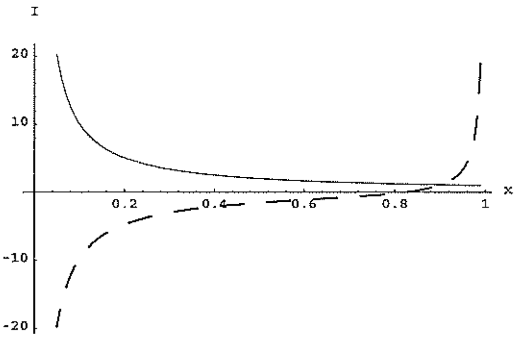

Figure 1. Steady state.

Figure 1 illustrates equations (35) and (37) in the (x,I)-space, where (35) is the dashed curve. For comparative dynamics we note that in terms of the composite parameters one has the implications: (i) $\Delta\mathcal{A} > 0 \Rightarrow \Delta\psi > 0 \Rightarrow$ both curves shift upward $\Rightarrow \Delta x < 0$, and (ii) $\Delta\mathcal{B} > 0 \Rightarrow$ curve (37) shifts upward $\Rightarrow \Delta x > 0$. Unfortunately, in spite of these observations the effects of changes in the parameters on inflation and growth remain often ambiguous, and specific numerical solutions must be sought.[11]

In Appendix 2 the feasibility of the model is investigated. It turns out that the general feasibility of the model can be stated in terms of a band, within which the term V/x must remain. We refer to Appendix 2 for details.

6. Basic Considerations for the Numerical Solution

In the next sections we will consider the main properties of the model. As already noted, unambiguous theoretical results are scarce and it is necessary to utilize numerical calculations for most of the results. One must thus specify numerical values for parameters, and below we select them in a way that is broadly in agreement with the literature and with some stylized facts of the US economy.

[11] *The numerical computations have been done using Mathematica. The routines are available upon request.*

The mean growth rate of productivity is fixed at 5.5 per cent throughout the experiments. This is broadly in line with the mean return on US stocks for the period 1929–89. The standard deviation of the productivity process is assumed to be 7.55 per cent. The mean and the standard deviation lead to output growth which broadly complies with the stylised facts of the U.S. economy for this period (with other parameter values discussed next).

Concerning the parameters of the utility function of the representative consumer it can be noted that in the literature there is relatively little agreement on the appropriate magnitudes: see Epstein and Zin (1991) and Obstfeld (1994a, b) for discussions and some numbers in the context of recursive preferences. We fix the subjective discount rate at 2 per cent per annum which corresponds to often-used low figures.[12] A benchmark value of 0.5 is postulated for the elasticity of intertemporal substitution (giving $z = -1$), but in some of the experiments the elasticity is allowed to vary in the range $(-1, -0.1)$.

We let the risk aversion parameter to take a wide range of values from .01 to 20 and compute the effects for this range in some of the experiments. $R = 5$ is used as a benchmark case. We are thus assuming in effect that the risk attitudes of the representative agent change from very risk-averse to very mildly risk-averse in the course of the experiment. It can be noted that in the literature estimates of risk aversion vary widely. For example, Lucas (1987, pp. 26) suggests the range 1–10, but he permits even the possibility 20 for relative risk aversion. Obstfeld (1994a) uses a similar range of values.

The reference case of a balanced primary budget assumes the expenditure and tax rates of 20 per cent, which is broadly in line with stylised facts after the transfer payments are netted out of the fiscal position of the public sector.[13] In policy experiments either expenditures are increased or taxes are cut to generate a permanent primary deficit. The base value of the bonds-to-money ratio f is assumed to be 3 which implies a 25 percent monetization of deficits. In some experiments f is increased to 6 which implies 1/7th fraction monetization of deficits.

7. Economic Policy and Growth

Let us now consider the growth and inflation effects of changes in the policy parameters in the complete model. The basic intuition is as follows. The general equilibrium condition for the inflation rate states that inflation rate is the difference of the rate of growth of money supply and the real growth rate of the economy. Then any policy measure that enhances real growth in the economy, *ceteris paribus,* slows down inflation. Thus changes in real growth represent in an intuitive sense changes in aggregate supply in the economy.

Policy measures have, of course, consequences to the financing structure of the government, thereby changing the money supply. The money supply effect then mars the above ceteris paribus link between policy changes, growth and inflation. It is clear in this model that, as was reported in a different risk-neutral context by Aiyagari and Gertler (1985), in general it matters considerably both in terms of the real and nominal implications whether the implied money supply change is caused by a change in the expenditure, tax, or open market policy.

Starting with growth, recall that the growth rate of real wealth was shown in (27) to follow a geometric Brownian motion. According to (27) growth depends on the expenditure parameter g through a direct crowding out effect and through the aggregate demand term V/x. Expenditure crowding out in this model is real in the sense that changes in government expenditures physically affect the flow of goods available for consumption and investment. All other policy measures instead have potential growth effects only through the aggregate demand term V/x.

The aggregate demand term depends on both the consumption-saving decisions (V) and the

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[12] *See e.g. Labadie (1989) and Epstein and Zin (1991).*

[13] *Lucas (1990) sets the value $g = .21$ for government consumption. His basic rate for capital taxation of 0.36 is higher than ours, but Lucas takes explicit account of government transfers which also include debt service (though he does not explicitly model debt). In a representative agent economy a first guess about transfers is that they are proportional to income, which in effect lowers the tax rate.*

risk taking decisions x of the representative agent. Fortunately, as can be seen in eq. (29), in equilibrium the two decisions are interlinked, so that larger V leads to smaller x and vice versa for a constant value for bonds-money ratio f. Then the aggregate demand effects of changes in risk aversion, government expenditure and taxation follow the qualitative pattern of the effects on V. After these observations it is in principle straightforward to analyze the effects of the different parameters on growth.

Looking first at the relation between uncertainty, risk aversion, intertemporal substitution and growth, numerical experiments indicate that most of the basic results obtained in the simple models of Sections 3.1 and 3.2 continue to hold in the presence of money, deficits and debt: higher risk aversion, and higher productivity shocks still lead to more rapid growth. As in the model without money (Section 3.2) the relation between the intertemporal substitution parameter z and mean growth is monotonically increasing for moderate values of R but it becomes monotonically decreasing when R is large enough. This result is qualitatively the same as in the simpler models in Section 3. Finally, with productivity shocks the (mean) growth rate is much less sensitive to the intertemporal substitution parameter z than under certainty.

Next we consider the growth effects of a change in either fiscal parameter g or s. For the expenditure ratio g it is seen from (27) that the stochastic part of the growth rate is simply the stochastic component of aggregate supply left after government purchases. Government expenditures have a trivial effect on the stochastic component: higher expenditure ratio results in smaller fluctuations in real variables.

For the benchmark values of $z = -1$ and $R = 5$, the aggregate demand effect of increased govt expenditures is actually growth enhancing. On the other hand, the crowding out effect slows down growth. The latter effect in fact dominates and the intended expansionary measure curbs real growth. However, the offsetting savings behaviour is fairly strong and the net effect of the measure is quite small. This conclusion holds for all values of z, but for variations in R it turns out that for high enough R increases in g can accelerate growth.[14]

A fall in the tax rate s only affects the growth rate through the aggregate demand term.[15] Analytically in terms of Figure 1 a cut in s implies *ceteris paribus* that the curve (37) shifts as a result of consequent change in $\mathcal{B}$ of eq. (37). The direction of the shift in $\mathcal{B}$ is in general uncertain (it depends on the sign of $r-R(1-g)\sigma^2$), so that the effect of a tax cut remains theoretically ambiguous.

A tax cut has the usual negative risk-smoothing effect in this model thereby reducing risk-taking and cutting back growth. However, it also accelerates inflation and pushes up the nominal interest rate in the general equilibrium. This induces the consumer to delay spending thereby accelerating growth, which tends to overcome in fact the risk-smoothing effect. Overall, a tax cut accelerates real growth, except in sufficiently risk-averse economies.

The quantitative magnitudes of the effects of changes in g and s are illustrated in panels A and B of Figure 2 giving the percentage rates of growth for the base case of $g = s = 0.2$ (solid lines) and alternative cases with either $g = 0.21$ or $s = 0.19$ (dashed lines). Growth rates are plotted as functions of R, since its magnitude influences the sign of the policy effect.

For changes in f via open-market operations we have a superneutrality result:

Proposition 2: The mean growth rate $(1-g)r-(V/x)$ is independent of f irrespective of the primary fiscal stance $g-s$.

Proof: Since g and r are exogenous constants the possible nonneutrality can only arise through the aggregate demand term V/x. Using (35) we have $V/x = \mathcal{A}/(xI-z)$. From (37) one obtains $xI-z = (xI-1)\mathcal{A}/\mathcal{B}$, so that $V/x = \mathcal{B}/(xI-1)$. Using (37) again we have $xI = (\mathcal{A}-\mathcal{B}z)/(\mathcal{A}-\mathcal{B})$. Since both $\mathcal{A}$ and $\mathcal{B}$ are independent of f, the result follows. ■

[14] *This result is in contrast to the corresponding result of the nonmonetary model.*

[15] *We note that, in contrast to the model in Section 3.2, the income tax rate influences the equilibrium directly and not only through the difference g–s. This is due to inflation and the assumed form of the tax system.*

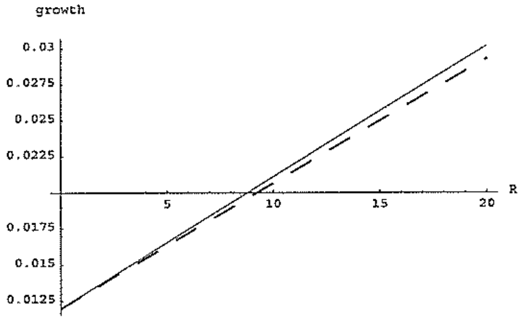

Figure 2A. g raised.

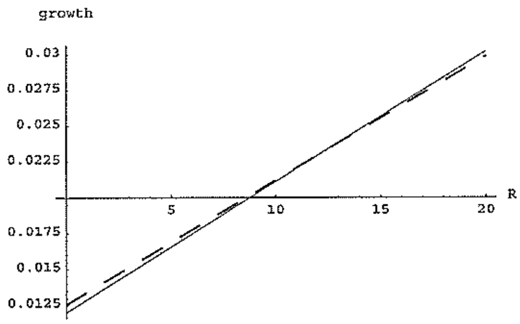

Figure 2B. s dropped.

Our neutrality result for the cash-in-advance model with endogenous growth extends related results in the literature. Danthine, Donaldson, and Smith (1987) showed in a similar model with money as an argument in an intertemporally and within-period separable utility function that money is superneutral under uncertainty. However, they only allowed for »helicopter drops» of money and considered neither money generation through budget deficits nor recursive non-expected utility preferences.

8. *Money, Inflation and Policy*

As noted in section 5 the effects of policy and parameter values on inflation are given by their influence on the difference between money and real growth. In thus remains to consider money growth, which can be derived using equations (25), (29) and (33). One obtains

$$(38)\quad dM/M = [(g-s)r(1-V)V^{-1} + fR\sigma_t^2(x_m-x) + fV/x]\,dt + (g-s)[(1+f)^{-1}V^{-1}-1]\sigma dz.$$

In (38) the deterministic part consists of three terms. The first term is generated by primary budget deviations. The other two terms are due to the deficits caused by interest payments, parts of which are financed by money issues. The second term is generated by the asset market risk premium, and the third is the aggregate demand term which bids up the nominal interest rate through its inflationary effect. The stochastic component of the money supply process is solely determined by primary budget deviations.

All the policy measures considered here (dg > 0, ds < 0, df > 0) are deficit-generating and, therefore, accelerate money growth. In the case of the change in bonds-money ratio f this verifies the property familiar from Sargent and Wallace (1981) unpleasant monetarist arithmetics analysis that open-market sales raise the total deficit flow and therefore increase the growth rates of both the money and bond stocks. The intuition is that interest bearing bonds are nominally more costly to the government than non-interest bearing money and therefore generate larger total deficits and higher inflationary pressures.

Combining the money supply effects with the growth effects in section 7 we get the price effects of policies. With the parameter values used in this analysis policy measures dg > 0 and ds < 0 df > 0 are inflationary, though the fiscal changes have an exception in very risk-averse economies. These results are illustrated by Figures 3 and 4. In Figure 3, panels A and B, we show the consequences on inflation of the fiscal policies considered here. In Figure 4 we provide a simulation of the geometric Brownian motions for inflation before and after the open market operation f = 3 ⇒ f = 6.[16] The significant inflationary impact of such a monetary policy is apparent.

Evaluating jointly the growth and inflation effects we conclude that, unless the degree of

[16] *We used the diffusion simulation program provided in Steele and Stine (1993).*

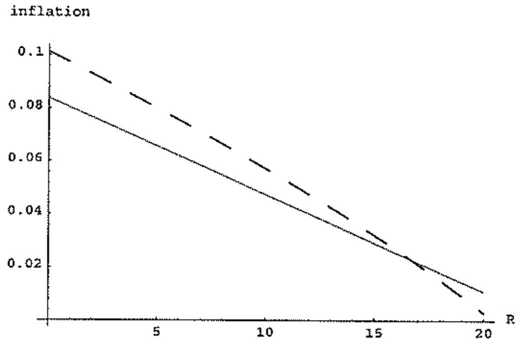

Figure 3A. g raised.

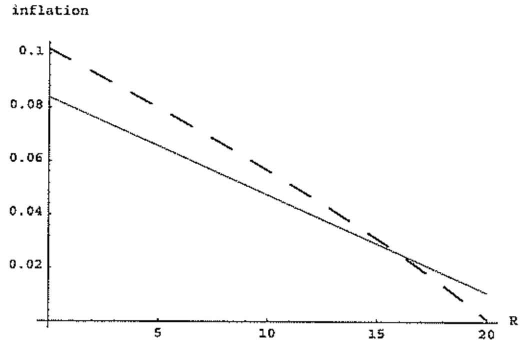

Figure 3B. s dropped.

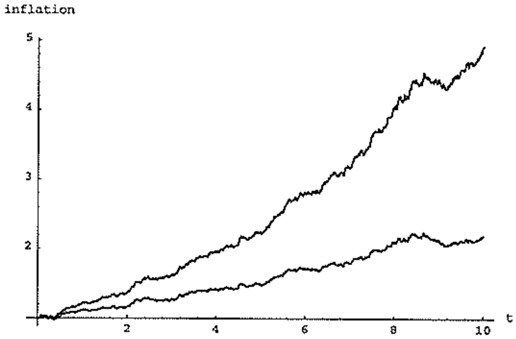

Figure 4.

risk aversion is very high, the Tobin–Mundell effect of an expenditure increase is reverse, i.e. reduction in growth and a rise in inflation will be observed simultaneously. Tax policy changes can produce either a standard or a reverse Tobin–Mundell effect in this model, depending on the parameter values. Our analysis thus shows that in a cash-in-advance economy the results are very different from those of Hartman (1987, 1988), who associated the sign of the Tobin-Mundell effect (in a stochastic Sidrauski-type model) with the magnitude of interest elasticity of the demand for money.

9. *Concluding Remarks*

In this paper we developed a simple framework of government activity, money and endogenous growth under uncertainty. The model operates in continuous time, but an exact solution taking the form of geometric Brownian motions for growth, inflation, and money growth was obtained. It would be important to try to generalize the model and obtain an exact solution without the important simplifying assumptions, especially about labor supply and production.

Recursive non-expected utility preferences permit the separation of intertemporal substitution and risk aversion in consumer preferences, so that their implications can be analyzed. We showed how the degree of risk aversion can be significant for the strength and even sign of the effects of certain government policies.

In our model the only source of uncertainty are the productivity shocks. It would be of interest to introduce monetary policy shocks in the model. Issues of permanent and temporary shocks and their role in the stochastic growth process could then be genuinely analysed. This remains a topic for further research.

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Appendix 1: Solution to Consumer's Optimization Problem

Here we provide details on the solution of the consumer's problem: maximize (1) subject to (18) and (19). We use the method of dynamic programming (see e.g. Merton 1971). We first adopt temporary notation $x_k = k/v$, $x_b = b/w$ and treat the cash-in-advance constraint (19) by introducing the constraint set

$$\text{(A.1)} \quad z = \{(w, C, x_k, x_b) \mid (1-x_k-x_b)w = C\}.$$

The value function in the dynamic programming is guessed to take the form $e^{-vt} J(w)$, where $J(w)$ is the current value function. The Hamiltin–Jacobi–Bellman equation then takes the form

$$\text{(A.2)} \quad 0 = -vF[(1-R)J(w)] + \max_z \{(1-R)C^z/z + (1-R)F'[(1-R)J(w)]\{J_w[[(1-s)x_k r + x_b((1-s)i-\pi_b) + (1-x_k-x_b)(-\pi_m)]w - C] + 2^{-1}J_{ww}[(1-s)^2\sigma^2 x_k^2 + (1-x_k)^2\sigma_p^2 - 2(1-s)x_k(1-x_k)\sigma_{kp}]w^2\}\}.$$

Differentiating (A.2) with respect to C, x_k and x_b gives respectively

$$\text{(A.3)} \quad (1-R)C^{z-1} - (1-R)F'[(1-R)J(w)]J_w - \lambda = 0,$$

$$(1-R)F'[(1-R)J(w)]\{wJ_w[(1-s)r + \pi_m] + w^2 J_{ww}[\sigma_t^2 - \sigma_p^2 - (1-s)\sigma_{kp}]\}$$

$$\text{(A.4)} \quad -\lambda w = 0,$$

$$\text{(A.5)} \quad (1-R)F[(1-R)J(w)]wJ_w[(1-s)i - \pi_b + \pi_m] - \lambda w = 0.$$

Here λ is the Lagrange multiplier associated with constraint (19) and $\sigma_t^2 = (1-s)^2\sigma^2 + 2(1-s)\sigma_{kp} + \sigma_p^2$. Next, one guesses that the value function is in fact of the form

$$\text{(A.6)} \quad J(w) = (Bw)^{1-R}/(1-R).$$

Solving λ from (A.5), and substituting it with J_w and J_{ww} from (A.6) into (A.3) and (A.4) leads to the results

$$C = wB^{z/(z-1)}[1 + (1-s)i - \pi_b + \pi_m]^{1/(z-1)} \equiv Vw,$$

$$x_k = \sigma_t^{-2}\{\sigma_p^2 + (1-s)\sigma_{kp} + R^{-1}[(1-s)(r-i) + \pi_b]\}.$$

Substituting these results back into (A.2) we get for the consumption propensity:

$$V = [v - zr_w + 2^{-1}zR\sigma_w^2]/[1 + (1-s)i - \pi_b + \pi_m - z].$$

Finally, the optimal choice of x_b can then be computed from the cash-in-advance constraint:

$$x_b = 1 - x_k - V.$$

This analysis justifies the system of equations (20)–(23) in the main text.

Appendix 2: Feasibility Constraints

The feasibility constraints deal with the implications of policies on the stock of government debt in the long run. Because of the fixed and finite bond-money ratio, the constraint can be stated in terms of the real money stock. The present value of real money (or bonds) should remain finite:

(A.7) $\lim_{t\to\infty} E_0[e^{-vt}m(t)] = 0.$

In addition, it is natural to require that the real money stock should not be expected to become zero, as this would not allow any consumption at all. This condition can be stated as

(A.8) $\lim_{t\to\infty} E_0[m(t)] > 0.$

To evaluate (A.7) and (A.8) we let

(A.9) $\bar{r}_g = (1-g)r - .5(1-g)^2\sigma^2$

denote the conditional expected growth rate of the economy. Since real money balances are proportional to real wealth, integrating (27) gives the following implicit constraints the policy parameters:[17]

(A.10) $\bar{r}_g > V/x > \bar{r}_g - v,$

where $\bar{r}_g$ is defined in (A.9). (A.10) is the most useful form for checking numerically the feasibility of different policy regimes.

Using the benchmark values for preference and production parameters we can illustrate numerically constraints (A.10) as a function of R and z. The dashed curves indicate the new curves after changing g and s to values g = .22 and s = .18. (In Figs A.1.1 and A.1.3 the bounds are computed using g = .22.) It is evident from the figures that different values for z do not cause problems with feasibility, while feasibility can be somewhat more delicate with sufficient risk aversion.

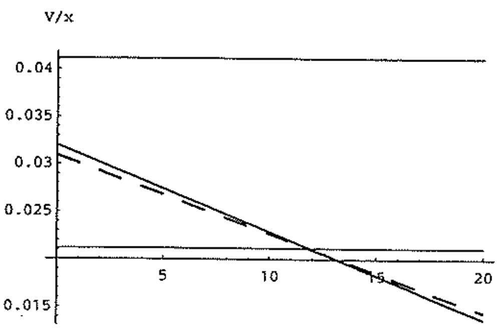

Figure A.1.1. Variation in g.

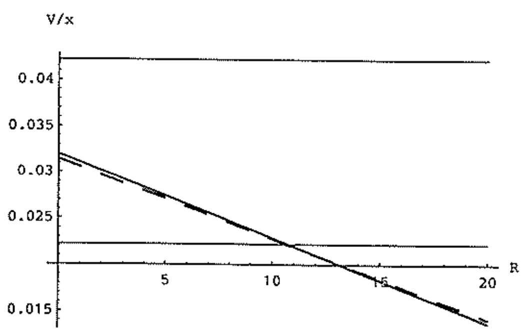

Figure A.1.2. Variation in s.

[17] *Since m follows a geometric Brownian motion, denoted temporarily as dm = m(αdt + βdz), letting N = logm the equation for dN can be integrated as in Gikhman & Skorohod (1972, pp. 33–39). Normalizing m(0) = 1 we get m(t) = exp[α–β²/2)t + βz(t)]. Thus (A.7) and (A.8) become α > β²/2 and α < v + β²/2.*

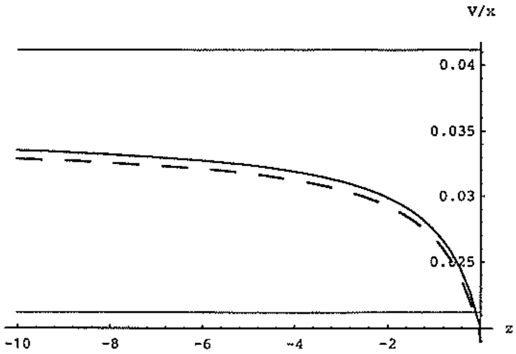

Figure A.1.3. Variation in g.

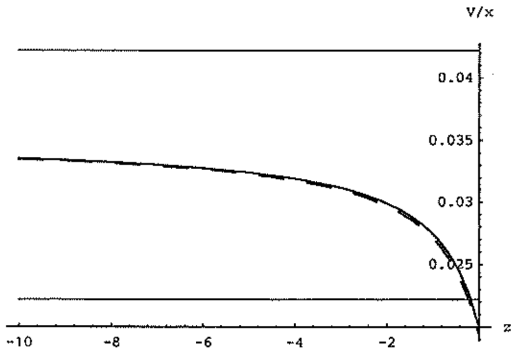

Figure A.1.4. Variation in s.